3/8/2011

Lecture 17

Quantum Typicality (ch. 14)

- need a notion of a quantum information source.
- Let it be some device that outputs random qudits according to some ensemble

$$\{p_Y(y), |\psi_y\rangle\}$$

- the density operator from the perspective of someone who doesn't know Y is

$$\rho = \mathbb{E}_Y\{|\psi_Y\rangle\langle\psi_Y|\} = \sum_y p(y)\, |\psi_y\rangle\langle\psi_y|$$

The spectral decomposition of ρ is

$$\sum_x p_X(x)\, |x\rangle\langle x|$$

where $\{|x\rangle\}$ is o.n. basis

can equivalently think of the source as emitting $\{p(x), |x\rangle\}$

3/8/2011

2

recall that $H(\rho) = H(X)$

Suppose that source emits many ~~realizations~~ quantum states:

$$\rho^{\otimes n} = \underbrace{\rho \otimes \rho \otimes \cdots \otimes \rho}_{n \text{ times}}$$

can write state as

$$\left(\sum_{x_1} p(x_1) \, |x_1\rangle\langle x_1|\right) \otimes \left(\sum_{x_2} p(x_2) \, |x_2\rangle\langle x_2|\right) \otimes \cdots \otimes \left(\sum_{x_n} p(x_n) \, |x_n\rangle\langle x_n|\right)$$

By linearity it is the same as

$$\sum_{x_1, x_2, \ldots, x_n} p(x_1) \, p(x_2) \cdots p(x_n) \, |x_1\rangle \cdots |x_n\rangle \langle x_1| \cdots \langle x_n|$$

$$\equiv \sum_{x^n \in \mathcal{X}^n} p(x^n) \, |x^n\rangle\langle x^n|^{X^n}$$

This is now looking remarkably similar to classical situation...

"Quantize" classical notion of typicality.

The δ-typical subspace $\mathcal{T}_\delta^{X^n}$ is associated to many copies of some density operator ρ.

Spanned by states whose classical sequences are typical:

$$\mathcal{T}_{\rho,\delta}^{X^n} \equiv \{ |x^n\rangle : x^n \in \mathcal{T}_\delta^{X^n} \}$$

$$= \{ |x^n\rangle : |\overline{H}(x^n) - H(X)| \leq \delta \}$$

Gives a way to divide up Hilbert space into two subspaces: typical + atypical

Typical Projector is very important:

$$\Pi_{\rho,\delta}^{X^n} = \sum_{x^n \in \mathcal{T}_\delta^{X^n}} |x^n\rangle\langle x^n|$$

can check that

$$\Pi_{\rho,\delta}^{X^n} \rho^{\otimes n} \Pi_{\rho,\delta}^{X^n} = \sum_{x^n \in \mathcal{T}_\delta^{X^n}} p(x^n) |x^n\rangle\langle x^n|$$

3/8/2011

can realize typical subspace measurement as a quantum instrument:

$$\sigma \rightarrow \left(I - \Pi_{\rho,\delta}^{X^n}\right) \sigma \left(I - \Pi_{\rho,\delta}^{X^n}\right) \otimes |0\rangle\langle 0| +$$
$$\Pi_{\rho,\delta}^{X^n} \sigma \Pi_{\rho,\delta}^{X^n} \otimes |1\rangle\langle 1|$$

very difficult to implement in practice.

Nevertheless, we proceed...

Typical Subspace has three properties

1) contains all the probability:

$$\mathrm{Tr}\left\{\Pi_{\rho,\delta}^{X^n} \rho^{\otimes n}\right\} \geq 1-\epsilon$$

Proof: $\mathrm{Tr}\left\{\Pi_{\rho,\delta}^{X^n} \rho^{\otimes n}\right\} =$

$$\mathrm{Tr}\left\{\Pi_{\rho,\delta}^{X^n} \rho^{\otimes n} \Pi_{\rho,\delta}^{X^n}\right\} =$$

$$\mathrm{Tr}\left\{\sum_{x^n \in T_\delta^{X^n}} p(x^n)\, |x^n\rangle\langle x^n|\right\}$$

$$= \sum_{x^n \in T_\delta^{X^n}} p(x^n) \geq 1-\epsilon$$

from classical typicality

3/8/2011

2) Exponentially small cardinality:

$$(1-\epsilon)2^{n[H(\rho)-\delta]} \leq \mathrm{Tr}\{\Pi_{\rho,\delta}^{X^n}\} \leq 2^{n[H(\rho)+\delta]}$$

Proof: $\mathrm{Tr}\{\Pi_{\rho,\delta}^{X^n}\}$

$$= \mathrm{Tr}\Big\{\sum_{x^n \in T_\delta^{X^n}} |x^n\rangle\langle x^n|\Big\}$$

$$= \sum_{x^n \in T_\delta^{X^n}} 1 = |T_\delta^{X^n}|$$

bounds then follow from classical proof.

3) Equipartition

$$2^{-n[H(\rho)+\delta]}\Pi_\delta^{X^n} \leq$$

$$\Pi_{\rho,\delta}^{X^n}\rho^{\otimes n}\Pi_{\rho,\delta}^{X^n} \leq 2^{-n[H(\rho)-\delta]}\Pi_{\rho,\delta}^{X^n}$$

Proof: $\Pi_{\rho,\delta}^{X^n}\rho^{\otimes n}\Pi_{\rho,\delta}^{X^n} = \sum_{x^n \in T^{X^n}} p(x^n)\,|x^n\rangle\langle x^n|$

eigenvalue bounds follow from classical typicality

3/8/2011

can now do the direct part of Schumacher's compression theorem (Ch. 17)

Task:

preparation: Source outputs a random sequence $|\psi_{x^n}\rangle \equiv |\psi_{x_1}\rangle \cdots |\psi_{x_n}\rangle$

- we don't know X^n, so we instead describe state w/ density operator

$$\rho^{\otimes n} = \rho \otimes \cdots \otimes \rho$$

Encoding: Alice performs some compression map $\mathcal{E}^{A^n \to W}$ where W is of size 2^{nR} (nR qubits)

Transmission: Alice transmits W to Bob w/ nR noiseless qubit channels

Decoding: Bob decompresses by sending through $\mathcal{D}^{W \to \hat{A}^n}$

protocol is good if

$$\left\| (\varphi_\rho^{RA})^{\otimes n} - (\mathcal{D}^{W \to \hat{A}^n} \circ \mathcal{E}^{A^n \to W})\left((\varphi^{RA})^{\otimes n}\right)\right\|_1 \leq \epsilon$$

rate R is achievable if there exists an (n, R, ϵ) compression code $\forall \epsilon > 0$ & sufficiently large n.

3/8/2011

Achievability (High Points)

There is state $\rho^{\otimes n}$ from source.

Encoding: Alice performs typical subspace measurement. succeeds w/ high probability

b/c $\mathrm{Tr}\{\Pi_{\rho,\delta}^{X^n}\rho^{\otimes n}\} \geq 1-\epsilon$

Also, post-measurement state

$$\frac{\Pi_{\rho,\delta}^{X^n}\rho^{\otimes n}\Pi_{\rho,\delta}^{X^n}}{\mathrm{Tr}\{\Pi_{\rho,\delta}^{X^n}\rho^{\otimes n}\}} \quad \text{is} \quad \sqrt{2\epsilon}\text{ - close}$$

to $\rho^{\otimes n}$. Acting on post-measurement state virtually the same as acting on $\rho^{\otimes n}$.

Classically, we had invertible compression map

$$f : T_\delta^{X^n} \to \{0,1\}^{n(H(X)+\delta)}$$

Turn this into isometry

$$\sum_{x^n \in T^{X^n}} |f(x^n)\rangle\langle x^n|$$

Alice applies + sends qubits over to Bob

3/8/2011

(8)

Bob receives + applies inverse isometry

$$\sum_{x^n \in \mathcal{T}^{X^n}_\delta} |x^n\rangle\langle f(x^n)|$$

getting $\frac{1}{\text{normalization}} \sum_{x^n \in \mathcal{T}^{X^n}_\delta} p(x^n)\, |x^n\rangle\langle x^n|$

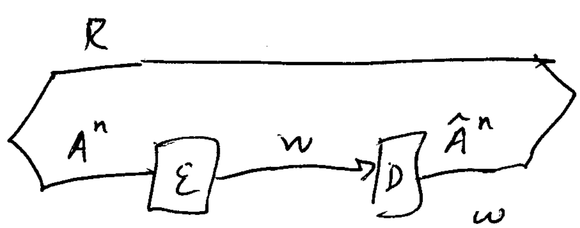
compression successful b/c this is approximately close to $\rho^{\otimes n}$

(everything would work as well on purifications b/c

$$\mathrm{Tr}\{\Pi^{X^n}_{\rho,\delta}\, \rho^{\otimes n}\} = \mathrm{Tr}\{(I^R \otimes \Pi^{X^n}_{\rho,\delta})(\varphi^{RX})^{\otimes n}\}$$

Converse Theorem

Recall



following criterion should hold for a good protocol

$$\|\omega^{R\hat{A}^n} - \varphi^{RA^n}\|_1 \le \epsilon$$

3|8|2011

$$
\begin{aligned}
2nR &= \log(2^{nR}) + \log(2^{nR}) \\
&\geq |H(W)| + |H(W|R)| \\
&\geq |H(W) - H(W|R)| \\
&= I(W;R) \\
&\geq I(\hat{A}^n;R)_\omega \qquad \text{(QDP)} \\
&\geq I(\hat{A};R)_\varphi - n\epsilon' \qquad \text{(Fannes')} \\
&= I(A;R)_\varphi - n\epsilon' \\
&= H(A)_\varphi + H(R)_\varphi - H(AR)_\varphi - n\epsilon' \\
&= H(A)_\varphi + H(A)_\varphi - n\epsilon' \\
&= 2H(A)_\varphi - n\epsilon'
\end{aligned}
$$

$\therefore$

$$R \geq H(A)_\rho - \epsilon'$$