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Lecture 15

Recall: quantum entropy

$$H(A)_\rho = -\mathrm{Tr}\{\rho \log \rho\}$$
$$= -\sum_x \lambda_x \log \lambda_x$$

where λ_x are eigenvalues of ρ

Suppose we have ensemble $\{p(x), |\psi_x\rangle\}$

ensemble has Shannon entropy $-\sum_x p(x) \log p(x)$

density operator is $\sum_x p(x) |\psi_x\rangle\langle\psi_x| \equiv \rho$

can prove that

$$H(\rho) \leq H(p(x))$$

Interpretation: requires less information to transmit in data compression

Example: $\left\{\left\{\frac{1}{4}, |0\rangle\right\}, \left\{\frac{1}{4}, |1\rangle\right\}, \left\{\frac{1}{4}, |+\rangle\right\}, \left\{\frac{1}{4}, |-\rangle\right\}\right\}$

Shannon entropy is 2 bits

Von Neumann is 1 bit

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Properties of Quantum Entropy

1) Positive
2) Minimum value is zero for pure state
3) Maximum value is $\log D$ where D is the dimension of system
4) Concavity:

if $\rho = \sum_x p(x) \rho_x$ then

$$H(\rho) \geq \sum_x p(x) H(\rho_x)$$

5) Unitary invariance:

$$H(\rho) = H(U \rho U^{\dagger})$$

(U does not change eigenvalues, analogous to permutation from classical case.)

Joint Quantum Entropy

$$H(AB)_\rho = -\mathrm{Tr}\{\rho^{AB} \log \rho^{AB}\}$$

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Striking property of quantum entropy

Classically, we always have

$$H(X,Y) \geq H(X) \quad \text{or}$$
$$H(X,Y) \geq H(Y)$$

(Adding another system can never decrease entropy.)

Quantumly, this does not necessarily have to happen

Suppose pure bipartite state $|\phi\rangle^{AB}$

Then $H(AB)_\phi = 0$ but

it holds that $H(A)_\phi = H(B)_\phi$

+ these can be > 0.

Proof Follows from Schmidt decomposition

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Quantum entropy is additive on tensor product states

$$H(\rho \otimes \sigma) = H(\rho) + H(\sigma)$$

Nice property   Implies that $H(\rho^{\otimes n}) = n H(\rho)$

Joint Entropy of a Classical-Quantum State

$\rho^{XB} = \sum_x p(x) |x\rangle\langle x|^X \otimes \rho_x^B$

What is $H(XB)_\rho$?

Claim: $H(XB)_\rho = H(X) + \sum_x p(x) H(\rho_x)$

(like the classical case)

Proof: diagonalize each ρ_x as

$$\rho_x^B = \sum_y p(y|x) |y_x\rangle\langle y_x|^B$$

Then $H(XB) = -\mathrm{Tr}\{\rho^{XB} \log \rho^{XB}\}$

$$= -\mathrm{Tr}\Big\{\Big(\sum_{x,y} p(x)\, p(y|x)\, |x\rangle\langle x| \otimes |y_x\rangle\langle y_x|\Big) \log\Big(\sum_{x',y'} p(x')\, p(y'|x')\, |x'\rangle\langle x'| \otimes |y'_{x'}\rangle\langle y'_{x'}|\Big)\Big\}$$

$$= -\mathrm{Tr}\Big\{\Big(\sum_{x,y} p(x)\, p(y|x)\, |x\rangle\langle x| \otimes |y_x\rangle\langle y_x|\Big) \sum_{x',y'} \log[p(x')\, p(y'|x')]\, |x'\rangle\langle x'| \otimes |y'_{x'}\rangle\langle y'_{x'}|\Big\}$$

$$= -\sum_{x,y,x',y'} p(x)\,p(y|x)\log\left(p(x')\,p(y'|x')\right) \operatorname{Tr}\{ |x\rangle\langle x|x'\rangle\langle x'| \otimes |y_x\rangle\langle y_x|y'_{x'}\rangle\langle y'_{x'}| \}$$

$$= -\sum_{x,y} p(x)\,p(y|x)\log\left(p(x)\,p(y|x)\right)$$

$$= -\sum_{x,y} p(x)\,p(y|x)\log\left(p(x)\right) - \sum_{x,y} p(x)\,p(y|x)\log\left(p(y|x)\right)$$

$$= H(X) + \sum_{x} p(x)\,H(\rho_x)$$

Conditional Quantum Entropy

$$H(A|B)_\rho \equiv H(AB)_\rho - H(B)_\rho$$

evaluated on ρ^{AB}

It holds that

$$H(A) \geq H(A|B)$$

"conditioning on a quantum system never increases entropy"

Conditional Entropy for a Classical-Quantum State

$\rho^{XB} = \sum_x p(x) |x\rangle\langle x|^X \otimes \rho_x^B$

$$H(B|X) = H(XB) - H(X)$$
$$= H(X) + \sum_x p(x) H(\rho_x) - H(X)$$
$$= \underbrace{\sum_x p(x) H(\rho_x)}$$

exactly like classical conditional entropy when conditioning system is classical

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Quantum Conditional Entropy can be negative!

Consider $H(A|B)$ on $|\Phi^+\rangle^{AB}$

$H(AB) = 0$

$H(B) = 1$ $\quad\quad$ $\mathrm{Tr}_A\{\Phi^{AB}\} = \pi^B$

$\therefore$ $H(A|B) = -1$

"can know more about the whole than the individual parts"

"the characteristic trait of quantum mechanics"
- Schrödinger

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negative of conditional entropy is very important. so important that we call it the "coherent information"

$$I(A\rangle B)_\rho = H(B)_\rho - H(AB)_\rho$$

$$I(A\rangle B)_\Phi = 1$$

Important for quantum capacity of particular channels

Can show that $I(A\rangle B) = H(A|E)$

for ρ^{AB} & $|\psi\rangle^{ABE}$ a purification of ρ

$$I(A\rangle B) = H(B) - H(AB) = H(AE) - H(E) = H(A|E)$$

can bound the absolute value of conditional entropy

$$|H(A|B)| \leq \log d_A$$

Proof $\quad H(A|B) \leq H(A) \leq \log d_A$

$$-H(A|B) = H(A|E) \leq H(A) \leq \log d_A$$

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Quantum Mutual Information

$$I(A;B) \equiv H(A) + H(B) - H(AB)$$
$$= H(A) + I(A\rangle B)$$
$$= H(B) + I(B\rangle A)$$

Theorem: $I(A;B) \geq 0$ (will prove later)

can prove that $I(A;B) \leq 2\min\{\log d_A, \log d_B\}$

can also show for $|\psi\rangle^{ABE}$

$$I(A\rangle B) = \tfrac{1}{2}I(A;B) - \tfrac{1}{2}I(A;E)$$

$$H(A) = \tfrac{1}{2}I(A;B) + \tfrac{1}{2}I(A;E)$$

(important relations)

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Conditional Quantum Mutual Information

$$I(A;B|C)_\rho = H(A|C)_\rho + H(B|C) - H(AB|C)_\rho$$

can prove $I(A;B|C)_\rho \geq 0$

(will do this later)

Quantum Relative Entropy

$$D(\rho\|\sigma) = \mathrm{Tr}\{\rho\log\rho - \rho\log\sigma\}$$

generalization of classical definition

measure of distinguishability, but not a metric.

Claim: $D(\rho\|\sigma) \geq 0$

Proof: Let $\rho = \sum_x p(x)\, |\phi_x\rangle\langle\phi_x|$

$\sigma = \sum_y q(y)\, |\psi_y\rangle\langle\psi_y|$

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$$\mathrm{Tr}\Big\{\sum_x p(x)\,|\phi_x\rangle\langle\phi_x|\Big[\log\Big(\sum_{x'} p(x')\,|\phi_{x'}\rangle\langle\phi_{x'}|\Big)$$
$$-\log\Big(\sum_y q(y)\,|\psi_y\rangle\langle\psi_y|\Big)\Big]\Big\}$$

$$= \mathrm{Tr}\Big\{\sum_x p(x)\,|\phi_x\rangle\langle\phi_x|\Big[\sum_{x'}\log p(x')\,|\phi_{x'}\rangle\langle\phi_{x'}|$$
$$-\sum_y \log(q(y))\,|\psi_y\rangle\langle\psi_y|\Big]\Big\}$$

$$= \sum_x p(x)\log p(x)$$

$$-\sum_{x,y} p(x)\log(q(y))\,\mathrm{Tr}\{|\phi_x\rangle\langle\phi_x|\psi_y\rangle\langle\psi_y|$$

$$= \sum_x p(x)\log(p(x))$$

$$-\sum_x p(x)\Big(\sum_y |\langle\phi_x|\psi_y\rangle|^2\log(q(y))\Big)$$

$$\geq \sum_x p(x)\log(p(x))$$

$$-\sum_x p(x)\log\Big(\sum_y |\langle\phi_x|\psi_y\rangle|^2 q(y)\Big)$$

$$\equiv \sum_x p(x)\log(p(x))$$

$$-\sum_x p(x)\log r(x)$$

$$= \sum_x p(x)\log\Big(\frac{p(x)}{r(x)}\Big) \geq 0$$

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Can show: $D(\rho^{AB} \| \rho^A \otimes \rho^B)$

$= I(A;B)_\rho$

$\therefore \quad I(A;B) \geq 0$

quantum relative entropy can go infinite

Let $\sigma \equiv \epsilon |\psi\rangle\langle\psi| + (1-\epsilon) |\psi^\perp\rangle\langle\psi^\perp|$

Then $\langle\psi|\sigma|\psi\rangle = \epsilon$

Consider

$$D(|\psi\rangle \| \sigma) = \mathrm{Tr}\{|\psi\rangle\langle\psi| \log |\psi\rangle\langle\psi| - |\psi\rangle\langle\psi| \log \sigma\}$$

$$= -\langle\psi| \log \sigma |\psi\rangle$$

$$= -\langle\psi| \left[\log(\epsilon) |\psi\rangle\langle\psi| + \log(1-\epsilon) |\psi^\perp\rangle\langle\psi^\perp|\right] |\psi\rangle$$

$$= -\log \epsilon$$

as $\epsilon \to 0 \quad D(|\psi\rangle \| \sigma) \to \infty$

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Fundamental Quantum Information Inequality

Monotonicity of Quantum Relative Entropy

$$D(\rho\|\sigma) \geq D(\mathcal{N}(\rho)\|\mathcal{N}(\sigma))$$

Simple (yet incomplete) proof

$$\begin{aligned}
D(\rho\|\sigma) &= D(\rho\|\sigma) + D(|0\rangle^E \| |0\rangle^E) \\
&= D(\rho \otimes |0\rangle\langle 0|^E \| \sigma \otimes |0\rangle\langle 0|^E) \\
&= D\left(U(\rho \otimes |0\rangle\langle 0|^E)U^\dagger \,\|\, U(\sigma \otimes |0\rangle\langle 0|^E) U^\dagger\right) \\
&\geq D(\mathcal{N}(\rho)\|\mathcal{N}(\sigma))
\end{aligned}$$

last inequality is simplified form where

$$D(\rho^{AB}\|\sigma^{AB}) \geq D(\rho^A\|\sigma^A)$$

difficult to prove & will not do so

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Important Inequality:

Complete Dephasing increases entropy

Consider $\overline{\Delta} = \sum_y |y\rangle\langle y| \cdot |y\rangle\langle y|$ can be different bases

Then suppose $\rho = \sum_x p(x)\, |x\rangle\langle x|$

& $\overline{\Delta}(\rho) = \sum_y |y\rangle\langle y| \left(\sum_x p(x)\,|x\rangle\langle x|\right) |y\rangle\langle y|$

$$= \sum_{y,x} p(x)\, |\langle y|x\rangle|^2\, |y\rangle\langle y|$$

$$= \sum_y \underbrace{\left(\sum_x p(x)\, |\langle y|x\rangle|^2\right)}_{\text{these are eigenvalues}} |y\rangle\langle y|$$

Would like to show that

$$H(\overline{\Delta}(\rho)) \geq H(\rho)$$

Consider $D(\rho \| \overline{\Delta}(\rho)) \geq 0$

$\therefore\ \mathrm{Tr}\{\rho \log \rho\} - \mathrm{Tr}\{\rho \log \overline{\Delta}(\rho)\} \geq 0$

$\therefore\ -H(\rho) - \mathrm{Tr}\left\{\sum_x p(x)\,|x\rangle\langle x| \left(\log \left(\sum_y \left(\sum_{x'} p(x')\,|\langle y|x'\rangle|^2\right) |y\rangle\langle y|\right)\right)\right\}$

$\therefore\ -H(\rho) - \mathrm{Tr}\left\{\sum_x p(x)\,|x\rangle\langle x| \left(\sum_y \log\left(\sum_{x'} p(x')\,|\langle y|x'\rangle|^2\right) |y\rangle\langle y|\right)\right\} \geq 0$

≥ 0

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$$\therefore\ -H(\rho) - \sum_{x,y} p(x) \log\left(\sum_{x'} p(x') |\langle y|x'\rangle|^2\right) \mathrm{Tr}\{|y\rangle\langle y| x\rangle\langle x|\} \geq 0$$

$$\therefore\ -H(\rho) - \sum_{y} \left(\sum_{x} p(x) |\langle y|x\rangle|^2\right) \log\left(\sum_{x'} p(x') |\langle y|x'\rangle|^2\right) \geq 0$$

$$\therefore\ -H(\rho) - H(\Delta(\rho)) \geq 0$$

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strong subadditivity follows from Monotonicity of QRE

consider that

$$D(\rho^{ABC} \| \pi^A \otimes \rho^{BC})$$

$$= \cancel{H(AB)}$$

$$-H(ABC)_\rho - \mathrm{Tr}\{\rho^{ABC} \log(\pi^A \otimes \rho^{BC})\}$$

$$= -H(ABC)_\rho - \mathrm{Tr}\{\rho^{ABC} \log(\pi^A \otimes I^{BC})\} - \mathrm{Tr}\{\rho^{ABC} \log(I^A \otimes \rho^{BC})\}$$

$$= -H(ABC)_\rho + \log d_A + H(BC)_\rho$$

$$= \log d_A - H(A|BC)_\rho$$

Similarly, $D(\rho^{AB} \| \pi^A \otimes \rho^B) = \log d_A - H(A|B)$

Then $D(\rho^{ABC} \| \pi^A \otimes \rho^{BC}) \geq D(\rho^{AB} \| \pi^A \otimes \rho^B) \iff H(A|B) \geq H(A|BC)$