Lecture 13

2/15/2011

Fidelity of two states

Begin w/ pure-state fidelity.

Suppose we input $|\psi\rangle$ into a protocol & it instead outputs $|\phi\rangle$:

Ideal: $|\psi\rangle^A \longrightarrow \boxed{I^{A\to B}} \longrightarrow |\psi\rangle^B$

Actual: $|\psi\rangle^A \longrightarrow \boxed{\text{Protocol}} \longrightarrow |\phi\rangle^B$

The pure-state fidelity is equal to the probability that $|\phi\rangle$ would pass a test for being $|\psi\rangle$:

$$F(|\psi\rangle, |\phi\rangle) \equiv |\langle\psi|\phi\rangle|^2$$

Thus, $0 \leq F(|\psi\rangle, |\phi\rangle) \leq 1$

0 if states are orthogonal &
1 if they are the same

Now suppose output of protocol is a mixed state ρ. Can think of ρ as arising from an ensemble $\{p(x), |\phi_x\rangle\}$

Then fidelity is average fidelity:

$$F(|\psi\rangle, \rho) = \mathbb{E}_X \{ |\langle\psi|\phi_X\rangle|^2 \}$$
$$= \sum_x p(x) \, |\langle\psi|\phi_x\rangle|^2$$
$$= \sum_x p(x) \, \langle\psi|\phi_x\rangle\langle\phi_x|\psi\rangle$$
$$= \langle\psi| \left(\sum_x p(x) |\phi_x\rangle\langle\phi_x| \right) |\psi\rangle$$
$$= \langle\psi|\rho|\psi\rangle$$

(It is the same regardless of the decomposition of ρ.)

$\langle\psi|\rho|\psi\rangle$ is 0 when support of ρ is orthogonal to $|\psi\rangle$

2/15/2011

Most general form of fidelity for two mixed states ρ^A + σ^A — borrow idea of pure-state fidelity as overlap of pure states, but instead take purifications $|\phi_\rho\rangle^{RA}$ + $|\phi_\sigma\rangle^{RA}$

$$F(\rho, \sigma) \equiv \max_{|\phi_\rho\rangle^{RA}, |\phi_\sigma\rangle^{RA}} |\langle \phi_\rho | \phi_\sigma \rangle|^2$$

all purifications are the same up to unitaries on the reference

$$\therefore \quad F(\rho, \sigma) = \max_{U_\rho^R, U_\sigma^R} |\langle \phi|_\rho \left((U_\rho^\dagger)^R \otimes I^A \right) (U_\sigma^R \otimes I^A) |\phi_\sigma\rangle|^2$$

$$= \max_{U_\rho^R, U_\sigma^R} |\langle \phi_\rho | (U_\rho^\dagger U_\sigma)^R \otimes I^A | \phi_\sigma \rangle|^2$$

$U_\rho^\dagger U_\sigma$ is just a single unitary, so

$$\therefore \quad F(\rho, \sigma) = \underbrace{\max_{U^R} |\langle \phi_\rho | U^R \otimes I^A | \phi_\sigma \rangle|^2}_{\text{Uhlmann fidelity}}$$

Uhlmann's Theorem

$$F(\rho,\sigma) = \max_U |\langle\phi_\rho| U^R \otimes I^A |\phi_\sigma\rangle^{RA}|^2$$
$$= \|\sqrt{\rho}\sqrt{\sigma}\|_1^2$$

Recall:
$\|A\|_1 = \mathrm{Tr}\{\sqrt{A^\dagger A}\}$

$\therefore \|\sqrt{\rho}\sqrt{\sigma}\|_1^2$
$= \mathrm{Tr}\{\sqrt{\sqrt{\rho}\sqrt{\sigma}\sqrt{\sigma}\sqrt{\rho}}\}^2$
$= \mathrm{Tr}\{\sqrt{\sqrt{\rho}\,\sigma\sqrt{\rho}}\}^2$

Proof: Suppose ρ has spectral decomposition

$$\rho = \sum_x p(x)\, |x\rangle\langle x|$$

can check that a purification of ρ is

$$|\phi_\rho\rangle \equiv \sqrt{d}\, (I^R \otimes \sqrt{\rho}^{\,X})\, |\Phi^+\rangle^{RX}$$

where $|\Phi^+\rangle^{RX} \equiv \frac{1}{\sqrt{d}} \sum_i |i\rangle^R |i\rangle^X$

Similarly,

$$|\phi_\sigma\rangle \equiv \sqrt{d}\, (I^R \otimes \sqrt{\sigma}^{\,X})\, |\Phi^+\rangle^{RX}$$

(holds regardless of basis of σ)

Let U^{*R} be the maximizing unitary

then

$$F(\rho,\sigma) = |\langle\phi_\rho| (U^*)^R \otimes I |\phi_\sigma\rangle|^2$$

5

2/15/2011

$$= \left| d \langle \Phi^+|^{RX} (U^*)^R \otimes \sqrt{\rho}^X \sqrt{\sigma}^X |\Phi^+\rangle^{RX} \right|^2$$

$$= \left| \sum_i \langle i|^R \langle i|^X (U^*)^R \otimes (\sqrt{\rho}\sqrt{\sigma})^X \sum_j |j\rangle^R |j\rangle^X \right|^2$$

$$= \left| \sum_{i,j} \langle i|^R \langle i|^X I^R \otimes (\sqrt{\rho}\sqrt{\sigma}(U^*)^T)^X |j\rangle^R |j\rangle^X \right|$$

$$= \left| \sum_{i,j} \langle i|j\rangle^R \langle i|^X (\sqrt{\rho}\sqrt{\sigma}(U^*)^T)^X |j\rangle^X \right|^2$$

$$= \left| \sum_i \langle i|^X \sqrt{\rho}\sqrt{\sigma}(U^*)^T |i\rangle^X \right|^2$$

$$= \left| \mathrm{Tr}\{\sqrt{\rho}\sqrt{\sigma}(U^*)^T\} \right|^2 \quad (*)$$

Aside: Every operator A has a "right polar decomposition"

$$A = \sqrt{AA^\dagger}\, V = |A|\, V$$

(Analogous to $z = re^{i\theta}$)

Proof: Follows from SVD

Suppose $A = U_1 D U_2$

$$\therefore \; AA^\dagger = U_1 D U_2 U_2^\dagger D U_1^\dagger$$

$$= U_1 D^2 U_1^\dagger$$

$$\therefore \; \sqrt{AA^\dagger} = U_1 D U_1^\dagger$$

can take V as $U_1 U_2$

~~matrix w/ phases where necessary~~

2/15/2011

Lemma: $|\mathrm{Tr}\{AU\}| \leq \mathrm{Tr}\{|A|\}$

w/ saturation when $U = V^\dagger$ where $A = |A|V$

Proof: $|\mathrm{Tr}\{AU\}| = |\mathrm{Tr}\{|A|VU\}|$

$= |\mathrm{Tr}\{|A|^{1/2}|A|^{1/2}VU\}|$ ← This is trace inner product $\mathrm{Tr}\{X^\dagger Y\}$ + we have $\mathrm{Tr}\{X^\dagger Y\} \leq \sqrt{\mathrm{Tr}\{X^\dagger X\}\,\mathrm{Tr}\{Y^\dagger Y\}}$

$\leq \sqrt{\mathrm{Tr}\{|A|\}\,\mathrm{Tr}\{U^\dagger V^\dagger |A| UV\}}$

$= \mathrm{Tr}\{|A|\}$

going back, (back to (*))

$|\mathrm{Tr}\{\sqrt{\rho}\sqrt{\sigma}\,(U^*)^T\}|^2$

$\leq \mathrm{Tr}\{|\sqrt{\rho}\sqrt{\sigma}|\}^2$

$= \|\sqrt{\rho}\sqrt{\sigma}\|_1^2$

maximizing unitary is from right polar decomposition of $\sqrt{\rho}\sqrt{\sigma}$ $\square$

2/15/2011

Properties of fidelity

Symmetry: $F(\rho, \sigma) = F(\sigma, \rho)$

evident from $\|\sqrt{\rho}\sqrt{\sigma}\|_1^2$ or

$$\max_{U^R} |\langle \phi_\rho | U^R | \phi_\sigma \rangle|^2$$

Monotonicity:

$$F(\rho^{AB}, \sigma^{AB}) \leq F(\rho^A, \sigma^A)$$

higher fidelity $\Leftrightarrow$ less distinguishable

Let $|\psi\rangle^{RAB}$ be a purification of ρ^{AB} & ρ^A

Let $|\phi\rangle^{RAB}$ be a purification of σ^{AB} & σ^A

then $$F(\rho^{AB}, \sigma^{AB}) = \max_{U^R \otimes I^{AB}} |\langle \psi|^{RAB} (U^R \otimes I^{AB}) |\phi\rangle^{RAB}|^2$$

$$F(\rho^A, \sigma^A) = \max_{U^{RB} \otimes I^A} |\langle \psi|^{RAB} (U^{RB} \otimes I^A) |\phi\rangle^{RAB}|$$

this maximization is inclusive of the prior one

Thus, $F(\rho^A, \sigma^A) \geq F(\rho^{AB}, \sigma^{AB})$

2/15/2011

Relationship between Trace Distance & Fidelity

$$1 - \sqrt{F(\rho,\sigma)} \leq \frac{1}{2} \|\rho - \sigma\|_1 \leq \sqrt{1 - F(\rho,\sigma)}$$

Suppose $F(\rho,\sigma) \geq 1 - \epsilon$ (two states are very similar)

Then $\epsilon \geq 1 - F(\rho,\sigma)$

$\therefore \quad \sqrt{\epsilon} \geq \sqrt{1 - F(\rho,\sigma)}$

$\therefore \quad \|\rho - \sigma\|_1 \leq 2\sqrt{\epsilon}$ (trace distance should be small)

Similarly, suppose $\|\rho - \sigma\|_1 \leq \epsilon$

$\therefore \quad 1 - \sqrt{F(\rho,\sigma)} \leq 1/2\, \epsilon$

$\therefore \quad \sqrt{F(\rho,\sigma)} \geq 1 - 1/2\, \epsilon$

$\therefore \quad F(\rho,\sigma) \geq (1 - 1/2\, \epsilon)^2$

$$= 1 - \epsilon + \frac{1}{4}\epsilon^2$$

$$\geq 1 - \epsilon$$

2/15/2011

Let's prove $\frac{1}{2}\|\rho-\sigma\|_1 \leq \sqrt{1-F(\rho,\sigma)}$

Consider ρ + σ pure

Suppose $|\psi\rangle$ + $|\phi\rangle$ where

$$|\phi\rangle = \cos(\theta)\,|\psi\rangle + e^{i\varphi}\sin\theta\,|\psi^{\perp}\rangle$$

where $|\psi^{\perp}\rangle = \underbrace{\left(I - |\psi\rangle\langle\psi|\right)|\phi\rangle}_{\text{normalization}}$

Fidelity is then

$$|\langle\phi|\psi\rangle|^2 = \cos^2\theta$$

consider $|\phi\rangle\langle\phi|$

$$|\phi\rangle\langle\phi| = \begin{bmatrix} \cos^2\theta & e^{-i\varphi}\sin\theta\cos\theta \\ e^{i\varphi}\sin\theta\cos\theta & \sin^2\theta \end{bmatrix}$$ in basis $\{|\psi\rangle, |\psi^{\perp}\rangle\}$

$\therefore$ $|\psi\rangle\langle\psi| - |\phi\rangle\langle\phi|$

$$= \begin{bmatrix} 1-\cos^2\theta & -e^{-i\varphi}\sin\theta\cos\theta \\ e^{i\varphi}\sin\theta\cos\theta & -\sin^2\theta \end{bmatrix}$$

eigenvalues are $|\sin\theta|$ + $-|\sin\theta|$

$\therefore$ $\|\,|\psi\rangle\langle\psi| - |\phi\rangle\langle\phi|\,\|_1 = 2|\sin\theta|$

2/15/2011

consider that

$$\left(\frac{2|\sin\theta|}{2}\right)^2 = 1 - \cos^2\theta$$

$$\therefore \left(\frac{\| |\psi\rangle\langle\psi| - |\phi\rangle\langle\phi| \|_1}{2}\right) = \sqrt{1 - F(|\psi\rangle, |\phi\rangle)}$$

To prove bound for mixed states ρ & σ, choose purifications $|\phi_\rho\rangle$ & $|\phi_\sigma\rangle$ such that

$$F(\rho^A, \sigma^A) = |\langle\phi_\sigma|\phi_\rho\rangle|^2 = F(|\phi_\rho\rangle^{RA}, |\phi_\sigma\rangle^{RA})$$

Then

$$\frac{1}{2}\|\rho^A - \sigma^A\|_1 \leq \frac{1}{2}\|\phi_\rho^{RA} - \phi_\sigma^{RA}\|_1$$

$$= \sqrt{1 - F(|\phi_\rho\rangle^{RA}, |\phi_\sigma\rangle^{RA})}$$

$$= \sqrt{1 - F(\rho^A, \sigma^A)}$$

2/15/2011

Application: Gentle Measurement

(form of information - disturbance trade-off)

Lemma: Consider ρ & Λ such that $0 \leq \Lambda \leq I$.

Suppose $\mathrm{Tr}\{\Lambda\rho\} \geq 1-\epsilon$ (*)
where $1 \geq \epsilon > 0$.

post-measurement state is

$$\rho' \equiv \frac{\sqrt{\Lambda}\rho\sqrt{\Lambda}}{\mathrm{Tr}\{\Lambda\rho\}}.$$

(*) implies that measurement barely changes the state, in the sense that

$$\|\rho - \rho'\|_1 \leq 2\sqrt{\epsilon}$$

Proof: First suppose ρ is pure $|\psi\rangle\langle\psi|$

Post-measurement state is

$$\frac{\sqrt{\Lambda}|\psi\rangle\langle\psi|\sqrt{\Lambda}}{\langle\psi|\Lambda|\psi\rangle}$$

Fidelity between this state and original is

$$\langle\psi|\left(\frac{\sqrt{\Lambda}|\psi\rangle\langle\psi|\sqrt{\Lambda}}{\langle\psi|\Lambda|\psi\rangle}\right)|\psi\rangle$$

2/15/2011

$$= \frac{|\langle\psi|\sqrt{\Lambda}|\psi\rangle|^2}{\langle\psi|\Lambda|\psi\rangle}$$

$$\geq \frac{|\langle\psi|\Lambda|\psi\rangle|^2}{\langle\psi|\Lambda|\psi\rangle} \qquad \left(\sqrt{\Lambda} \geq \Lambda \text{ when } \Lambda \leq I\right)$$

$$= \langle\psi|\Lambda|\psi\rangle$$

$$= \mathrm{Tr}\{\Lambda\, |\psi\rangle\langle\psi|\} \geq 1-\epsilon$$

Now consider mixed states

ρ^A + ρ'^A Let $|\psi\rangle^{RA}$ + $|\psi'\rangle^{RA}$ be purificat

$$|\psi'\rangle^{RA} = \frac{I^R \otimes \sqrt{\Lambda}^A |\psi\rangle^{RA}}{\langle\psi|^{RA} I^R \otimes \Lambda^A |\psi\rangle^{RA}}$$

$$\therefore \quad F(\rho^A, \rho'^A) \geq F(|\psi\rangle^{RA}, |\psi'\rangle^{RA}) \quad \text{(monotonicity)}$$

$$\geq 1-\epsilon$$

$\|\rho^A - \rho'^A\| \leq 2\sqrt{\epsilon}$ by relationship between trace distance + fidelity. $\square$

there are other useful variations of this lemma