Lecture 12

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Distance Measures

- Protocols considered so far have been noiseless (though, we developed noisy quantum theory)

- we would like ways of determining how close an imperfect protocol is to being perfect:

$$|\psi\rangle^A \longrightarrow \boxed{I^{A\to B}} \longrightarrow |\psi\rangle^B \quad \text{vs.}$$

$$|\psi\rangle^A \longrightarrow \boxed{\mathcal{N}^{A\to B}} \longrightarrow \rho^B$$

$$D(\psi, \rho) \leq \epsilon$$

need a measure of distance

two main measures: trace distance and fidelity

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Trace distance arises from the trace norm of an operator M is

$$\|M\|_1 = Tr\left\{\sqrt{M^\dagger M}\right\}$$

just consider M Hermitian

then $M = \sum_i \mu_i |i\rangle\langle i|$

$$\Rightarrow M^\dagger M = \sum_i |\mu_i|^2 |i\rangle\langle i|$$

$$\& \sqrt{M^\dagger M} = \sum_i \sqrt{|\mu_i|^2}\, |i\rangle\langle i|$$

$$= \sum_i |\mu_i|\, |i\rangle\langle i|$$

$$\Rightarrow Tr\left\{\sqrt{M^\dagger M}\right\} = Tr\left\{\sum_i |\mu_i|\, |i\rangle\langle i|\right\}$$

$$= \sum_i |\mu_i|\, Tr\{|i\rangle\langle i|\}$$

$$= \sum_i |\mu_i|$$

absolute sum of eigenvalues

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5 useful properties for trace distance

1) positive definiteness: $\|M\|_1 \geq 0$

$+ \quad \|M\|_1 = 0 \iff M = 0$

2) Homogeneity: For $c \in \mathbb{C}$

$$\|cM\|_1 = |c| \, \|M\|_1$$

3) Triangle Inequality: (important for bounding errors)

$$\|M + N\|_1 \leq \|M\|_1 + \|N\|_1$$

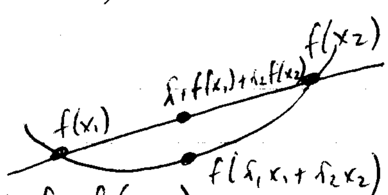
4) Isometric Invariance:

$$\|UMU^{+}\|_1 = \|M\|_1 \qquad \text{(why?)}$$

5) Convexity (holds for any norm):

$$\|\lambda_1 M + \lambda_2 N\|_1 \leq \lambda_1 \|M\|_1 + \lambda_2 \|N\|_1$$

where $\lambda_1, \lambda_2 \geq 0$ + $\lambda_1 + \lambda_2 = 1$



convex smile ☺

$$f(\lambda_1 x_1 + \lambda_2 x_2) \leq \lambda_1 f(x_1) + \lambda_2 f(x_2)$$

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Trace norm induces trace distance:

$$\|M-N\|_1$$

useful as measure of distinguishability for density operators ρ & σ

ρ & σ are equal if $\|\rho-\sigma\|_1 = 0$

& an upper bound is

$$\|\rho-\sigma\|_1 = \|\rho + (-\sigma)\|_1$$
$$\leq \|\rho\|_1 + \|\sigma\|_1 = 1+1=2$$

Trace distance saturated when ρ & σ are on orthogonal subspaces

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Different characterization of trace distance:

$$\|\rho-\sigma\|_1 = 2 \max_{0 \leq \Lambda \leq I} \mathrm{Tr}\{\Lambda(\rho-\sigma)\}$$

think of Λ as part of measurement used to distinguish ρ from σ (will see this later)

Proof:

$\rho-\sigma$ Hermitian

$$\Rightarrow \rho-\sigma = U D U^{\dagger}$$
$$= U(D^{+}-D^{-})U^{\dagger}$$
$$= \underbrace{U D^{+} U^{\dagger}}_{\alpha^{+}} - \underbrace{U D^{-} U^{\dagger}}_{\alpha^{-}}$$

Let Π^{+} be projector onto α^{+} & Π^{-} projector onto α^{-}

then

$$|\rho-\sigma| = \alpha^{+} + \alpha^{-}$$

So
$$\|\rho-\sigma\|_1 = \mathrm{Tr}\{|\rho-\sigma|\}$$
$$= \mathrm{Tr}\{\alpha^{+}+\alpha^{-}\}$$
$$= \mathrm{Tr}\{\alpha^{+}\} + \mathrm{Tr}\{\alpha^{-}\}$$

But $\mathrm{Tr}\{\alpha^+\} - \mathrm{Tr}\{\alpha^-\}$

$$= \mathrm{Tr}\{\alpha^+ - \alpha^-\}$$

$$= \mathrm{Tr}\{\rho - \sigma\} = 0$$

$\therefore\ \mathrm{Tr}\{\alpha^+\} = \mathrm{Tr}\{\alpha^-\}$

So, $\|\rho - \sigma\|_1 = 2\,\mathrm{Tr}\{\alpha^+\}$

Now consider

$$2\,\mathrm{Tr}\{\Pi^+(\rho-\sigma)\} = 2\,\mathrm{Tr}\{\Pi^+(\alpha^+ - \alpha^-)\}$$
$$= 2\,\mathrm{Tr}\{\Pi^+\alpha^+\}$$
$$= 2\,\mathrm{Tr}\{\alpha^+\}$$
$$= \|\rho-\sigma\|_1$$

prove that Π^+ is maximizing operator:

$$2\,\mathrm{Tr}\{\Lambda(\rho-\sigma)\} = 2\,\mathrm{Tr}\{\Lambda(\alpha^+ - \alpha^-)\}$$
$$\leq 2\,\mathrm{Tr}\{\Lambda\alpha^+\}$$
$$\leq 2\,\mathrm{Tr}\{\alpha^+\}$$
$$= \|\rho-\sigma\|_1$$

Operational Interpretation of Trace Distance (sets it on firm footing)

Hypothesis testing – prepare ρ_0 or ρ_1 w/ equal probability

$$p_X(0) = p_X(1) = 1/2$$

Bob doesn't know which one prepared & must perform a measurement to distinguish – POVM $\{\Lambda_0, \Lambda_1\}$

$\Lambda_0 \rightarrow$ guess state was ρ_0
$\Lambda_1 \rightarrow$ guess state was ρ_1

probability of error is sum of false positive + false negative probabilities

$$p_e = p(0|1)\, p_X(1) + p(1|0)\, p_X(0)$$
$$= \mathrm{Tr}\{\Lambda_0 \rho_1\} 1/2 + \mathrm{Tr}\{\Lambda_1 \rho_0\} 1/2$$

using the fact that $\Lambda_0 + \Lambda_1 = I$, can rewrite p_e

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$$p_e = \frac{1}{2}\left[\mathrm{Tr}\{\Lambda_0 \rho_1\} + \mathrm{Tr}\{\Lambda_1 \rho_0\}\right]$$
$$= \frac{1}{2}\left\{\mathrm{Tr}\{\Lambda_0 \rho_1\} + \mathrm{Tr}\{(I - \Lambda_0)\rho_0\}\right\}$$
$$= \frac{1}{2}\left\{\mathrm{Tr}\{\rho_0\} + \mathrm{Tr}\{\Lambda_0(\rho_1 - \rho_0)\}\right\}$$
$$= \frac{1}{4}\left\{2 - 2\,\mathrm{Tr}\{\Lambda_0(\rho_0 - \rho_1)\}\right\}$$

Bob has freedom in choosing POVM.
Thus,

$$p_e^* = \min_{\{\Lambda_0, \Lambda_1\}} p_e$$
$$= \frac{1}{4}\left\{2 - \max_{\{\Lambda_0, \Lambda_1\}} 2\,\mathrm{Tr}\{\Lambda_0(\rho_0 - \rho_1)\}\right\}$$
$$= \frac{1}{4}\left[2 - \|\rho_0 - \rho_1\|_1\right]$$

if $\rho_0 = \rho_1 \Rightarrow p_e^* = 1/2$ (just as good as a random guess)

if $\|\rho_0 - \rho_1\|_1 = \underset{\text{maximum}}{2} \Rightarrow p_e^* = 0$ perfectly distinguishable

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Lemmas

Triangle Inequality (helpful for bounding errors)

$$\|\rho-\sigma\|_1 \leq \|\rho-\tau\|_1 + \|\tau-\sigma\|_1$$

Proof.

$$\begin{aligned}\|\rho-\sigma\|_1 &= 2\,\mathrm{Tr}\{\Pi^*(\rho-\sigma)\} \\ &= 2\,\mathrm{Tr}\{\Pi^*(\rho-\tau)\} + \\ &\quad 2\,\mathrm{Tr}\{\Pi^*(\tau-\sigma)\} \\ &\leq \|\rho-\tau\|_1 + \|\tau-\sigma\|_1\end{aligned}$$

b/c Π^* maybe not optimal for distinguishing ρ from τ & τ from σ

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Measurement on approximately close states

suppose Π : $0 \leq \Pi \leq I$

Then $\mathrm{Tr}\{\Pi\rho\} \geq \mathrm{Tr}\{\Pi\sigma\} - \|\rho-\sigma\|_1$

Most common use:
measurement successful on σ : $\mathrm{Tr}\{\Pi\sigma\} \geq 1-\epsilon$

ρ close to σ : $\|\rho-\sigma\|_1 \leq \epsilon$

then measurement should be successful on ρ :

$$\mathrm{Tr}\{\Pi\rho\} \geq 1-2\epsilon$$

Proof: $\|\rho-\sigma\|_1 = 2 \max_{0\leq\Lambda\leq I} \mathrm{Tr}\{\Lambda(\sigma-\rho)\}$

$$\geq \mathrm{Tr}\{\Pi(\sigma-\rho)\}$$

$$= \mathrm{Tr}\{\Pi\sigma\} - \mathrm{Tr}\{\Pi\rho\}$$

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Monotonicity

states become less distinguishable after discarding;

$$\|\rho^A - \sigma^A\|_1 \leq \|\rho^{AB} - \sigma^{AB}\|_1$$

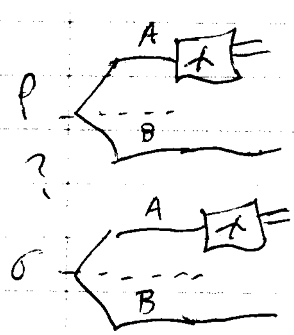
Proof:

Suppose $\|\rho^A - \sigma^A\|_1 = 2\mathrm{Tr}\{\Lambda^A(\rho^A - \sigma^A$ for some Λ^A

then

$$\begin{aligned}
\|\rho^A - \sigma^A\|_1 &= 2\mathrm{Tr}\{\Lambda^A(\rho^A - \sigma^A)\} \\
&= 2\mathrm{Tr}\{(\Lambda^A \otimes I^B)(\rho^{AB} - \sigma^{AB})\} \\
&\leq 2 \max_{0 \leq \Lambda^{AB} \leq I} \mathrm{Tr}\{\Lambda^{AB}(\rho^{AB} - \sigma^{AB}\} \\
&= \|\rho^{AB} - \sigma^{AB}\|_1
\end{aligned}$$

Think hypothesis testing:



is worse than