Lecture 11

2/8/2011

Recall teleportation

$$2[c \to c] + [qq] \geq [q \to q]$$

& dense coding

$$[q \to q] + [qq] \geq 2[c \to c]$$

asymmetric (not dual under resource reversal)

If entanglement free, then they are

$$2[c \to c] = [q \to q]$$

Better way to make them dual under resource reversal is w/ a coherent bit channel

Coherent Communication

Recall that a classical channel is the same as a completely dephasing channel:

$$\Delta(\rho) = \frac{1}{2}(\rho + Z\rho Z) = |0\rangle\langle 0|\rho|0\rangle\langle 0| + |1\rangle\langle 1|\rho|1\rangle\langle 1|$$

2/8/2011

isometric extension is

$$U_{\mathcal{N}}^{A\to BE} = |0\rangle^B\langle 0|^A \otimes |0\rangle^E + |1\rangle^B\langle 1|^A \otimes |1\rangle^E$$

equivalent to the map

$$|i\rangle^A \to |i\rangle^B |i\rangle^E \quad i \in \{0,1\}$$

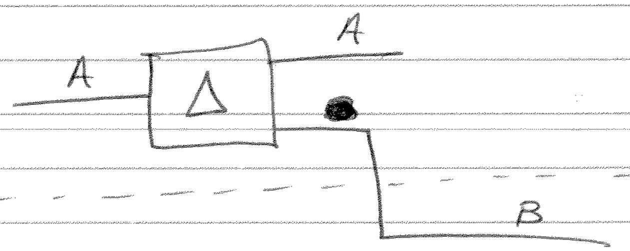
Suppose Alice can regain control of the environment somehow, so that map is

$$|i\rangle^A \to |i\rangle^B |i\rangle^A$$

(some preferred basis)

"quantum feedback" channel

write as



denote resource as $[q \to qq]$

can show that

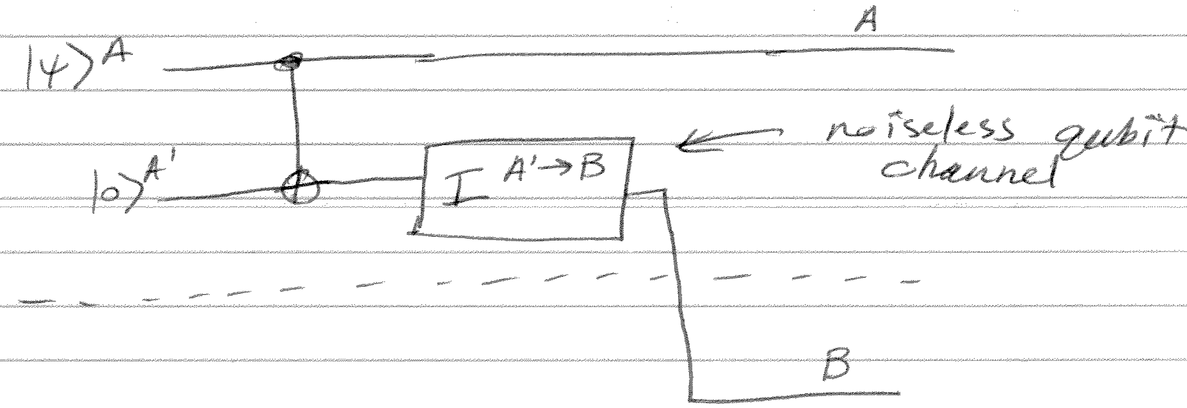
$$[q\to qq] \geq [c \to c]$$
$$[q \to qq] \geq [qq]$$

(but the other ways do not hold)

2/8/2011

How to implement a coherent bit channel?

simplest way:



$$|\psi\rangle^A|0\rangle^{A'} = \alpha|0\rangle^A|0\rangle^{A'} + \beta|1\rangle^A|0\rangle^{A'}$$

$$\xrightarrow{\text{CNOT}} \alpha|0\rangle^A|0\rangle^{A'} + \beta|1\rangle^A|1\rangle^{A'}$$

$$\xrightarrow{I^{A'\to B}} \alpha|0\rangle^A|0\rangle^B + \beta|1\rangle^A|1\rangle^B$$

$$[q\to q] \geq [q\to qq]$$

(there are smarter ways to implement)

2/8/2011 (4)

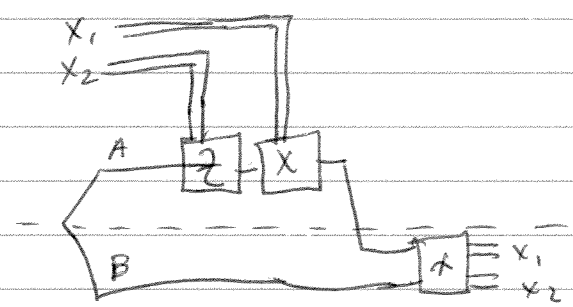
so, we have

$$[q \to q] \geq [q \to qq] \geq [qq]$$

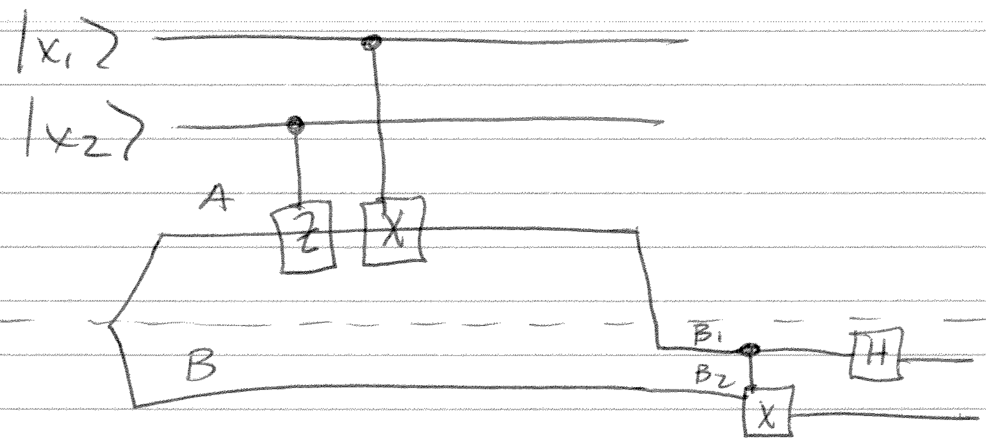
(noiseless qubit) (coherent bit) (ebit)

Coherent Dense Coding

Recall dense coding



do every step coherently instead



2/8/2011

Why does this work?

$$|x_1\rangle^{A_1} |x_2\rangle^{A_2} \quad |\Phi^+\rangle^{AB}$$

$$\longrightarrow \quad |x_1\rangle^{A_1} |x_2\rangle^{A_2} \quad \left(Z^{x_2} X^{x_1}\right)^{A} |\Phi^+\rangle^{AB}$$

$$\xrightarrow{\text{CNOT} \,\&\, H} \quad |x_1\rangle^{A_1} |x_2\rangle^{A_2} \qquad |x_2\rangle^{B_2} |x_1\rangle^{B_1}$$

implements 2 cobit channels

one from

$$|x_1\rangle^{A_1} \longrightarrow |x_1\rangle^{A_1} |x_1\rangle^{B_1}$$

& another

$$|x_2\rangle^{A_2} \longrightarrow |x_2\rangle^{A_2} |x_2\rangle^{B_2}$$

can do everything in ~~s~~ superposition...

Thus, we have

$$[q \to q] + [qq] \geq 2 [q \to qq]$$

stronger than dense coding

can do this w/ qudits as well

2/8/2011

Coherent Teleportation

define Δ_Z as $|0\rangle^A \to |0\rangle^A |0\rangle^B$

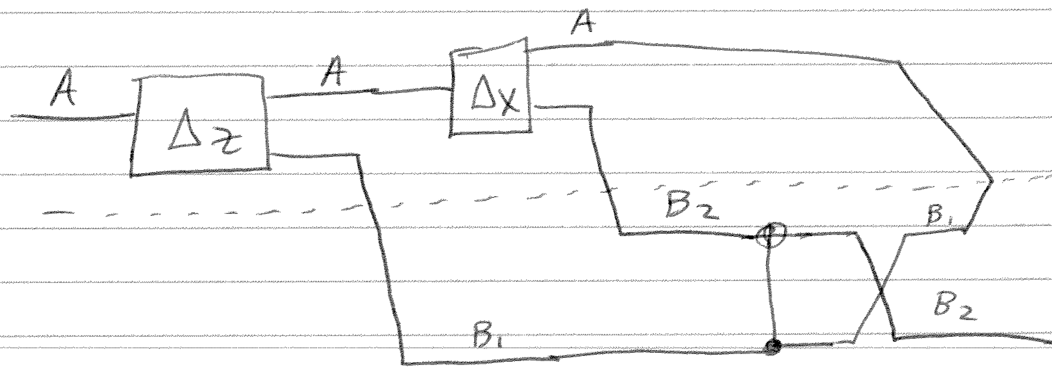
$|1\rangle^A \to |1\rangle^A |1\rangle^B$

+ Δ_X as $|+\rangle^A \to |+\rangle^A |+\rangle^B$

$|-\rangle^A \to |-\rangle^A |-\rangle^B$

can simulate one w/ the other and local ops

Picture is



Why does this work?

1) $|\psi\rangle^A = \alpha|0\rangle^A + \beta|1\rangle^A$

$\xrightarrow{\Delta_Z} \alpha|0\rangle^A|0\rangle^{B_1} + \beta|1\rangle^A|1\rangle^{B_1}$

rewrite state as

$$\alpha\left(\frac{|+\rangle^A + |-\rangle^A}{\sqrt{2}}\right)|0\rangle^{B_1} + \beta\left(\frac{|+\rangle^A - |-\rangle^A}{\sqrt{2}}\right)|1\rangle^{B_1}$$

2/8/2011 (7)

$$= \frac{1}{\sqrt{2}}\Big[|+\rangle^A \left(\alpha|0\rangle^{B_1} + \beta|1\rangle^{B_1}\right) + |-\rangle^A \left(\alpha|0\rangle^{B_1} - \beta|1\rangle^{B_1}\right)\Big]$$

send A through X cobit

$$\rightarrow \frac{1}{\sqrt{2}}\Big[|+\rangle^A|+\rangle^{B_2}\left(\alpha|0\rangle^{B_1} + \beta|1\rangle^{B_1}\right) + |-\rangle^A|-\rangle^{B_2}\left(\alpha|0\rangle^{B_1} - \beta|1\rangle^{B_1}\right)\Big]$$

need to eliminate phase

CNOT from B_1 to B_2

$$\rightarrow \frac{1}{\sqrt{2}}\left(|+\rangle^A|+\rangle^{B_2} + |-\rangle^A|-\rangle^{B_2}\right)\left(\alpha|0\rangle^{B_1} + \beta|1\rangle^{B_1}\right)$$

$$= |\Phi^+\rangle^{AB_2}\,|\psi\rangle^{B_1}$$

entanglement generation   teleportation

$$2[q \to qq] \geq [q \to q] + [qq]$$

(can also do this by modifying original teleportation protocol — replace Bell measurements + conditional operations w/ cobit channels of controlled operations)

2/8/2011

Coherent Comm. Identity

$$2\{q\to qq\} = \{q\to q\} + \{qq\}$$

Why is this important? beautiful + practical

Suppose there is some protocol

$\langle \mathcal{N}\rangle + E\{qq\} \geq R\{c\to c\}$

such that classical bits are private from the environment of the channel

then we can upgrade classical bits to coherent bits:

$\langle \mathcal{N}\rangle + E\{qq\} \geq R\{q\to qq\}$

exploit coherent comm. identity:

$\langle \mathcal{N}\rangle + E\{qq\} \geq \frac{R}{2}\left(\{q\to q\} + \{qq\}\right)$

$\therefore \langle \mathcal{N}\rangle + E\{qq\} \geq \frac{R}{2}\{q\to q\} + \frac{R}{2}\{qq\}$

consider catalytic use of entanglement

$\therefore \langle \mathcal{N}\rangle + \left(E - \frac{R}{2}\right)\{qq\} \geq \frac{R}{2}\{q\to q\}$

Unit Resource Capacity Region

What is the best we can do w/ noiseless resources of q. comm., c. comm., + entanglement?

We know

TP: $2\{c\to c\} + \{qq\} \geq \{q\to q\}$

SD: $\{q\to q\} + \{qq\} \geq 2\{c\to c\}$

ED: $\{q\to q\} \geq \{qq\}$

Consider these as points in a 3D region:

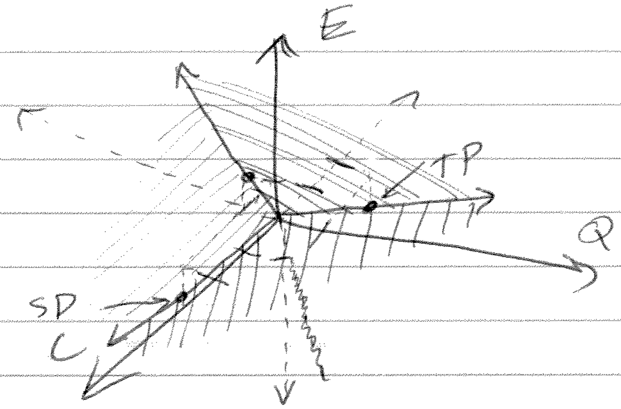
(C, Q, E)

TP is $(-2, 1, -1) \equiv x_{TP}$

SD is $(2, -1, -1) \equiv x_{SD}$

ED is $(0, -1, 1) \equiv x_{ED}$

2/8/2011



can achieve any point

$(C, Q, E) = \alpha\, x_{TP} + \beta\, x_{SD} + \gamma\, x_{ED}$ where $\alpha, \beta, \gamma \geq 0$

equivalent to matrix relation

$$\begin{bmatrix} C \\ Q \\ E \end{bmatrix} = \begin{array}{ccc} & & \\ \left[\begin{array}{ccc} -2 & 2 & 0 \\ 1 & -1 & -1 \\ -1 & -1 & 1 \end{array}\right] \\ \uparrow \quad \uparrow \quad \uparrow \\ TP \quad SD \quad ED \end{array} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$$

Equivalent to

$$\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} -1/2 & -1/2 & -1/2 \\ 0 & -1/2 & -1/2 \\ -1/2 & -1 & 0 \end{bmatrix} \begin{bmatrix} C \\ Q \\ E \end{bmatrix}$$

matrix inverse

2/8/2011

Restriction that $\alpha, \beta, \gamma \geq 0$

$$\Rightarrow \quad C + Q + E \leq 0$$
$$Q + E \leq 0$$
$$C + 2Q \leq 0$$

this region is achievable.

Is it optimal? (converse theorem)

consider eight octants individually to prove

take as given 1) ~~classical~~ $\{c \to c\} \not\geq Q\{q \to q\}$

$\{c \to c\} \not\geq E\{qq\}$

2) $\{qq\} \not\geq Q\{q \to q\}$

$\{qq\} \not\geq C\{c \to c\}$

Consider octant $(+, +, +)$

must be empty b/c we require consumption of resources to generate others

octant $(-, +, +)$

empty b/c 1)

octant $(+, +, -)$ empty b/c 2)

2/8/2011 (12)

do 2 other octants

(+, −, +) task is to generate C classical bits + E ebits using $|Q|$ qubits

consider points of the form

(C, Q, E) where $C \geq 0$, $Q \leq 0$, + $E \geq 0$

just prove that $C + E \leq |Q|$
b/c combining w/ $C \geq 0$ + $E \geq 0$
gives

$C + E \leq |Q|$
$E \leq |Q|$ (redundant)
$C \leq 2|Q|$ (redundant

Suppose $(C, -|Q|, E)$ exists, can then combine w/ dense coding $(2E, -E, -E)$

$$(C, -|Q|, E) + (2E, -E, -E) =$$

$$\longrightarrow (C + 2E, -|Q| - E, 0)$$

this is protocol generates classical comm. from qu. com
had better respect Holevo bound $C + 2E \leq |Q| + E$

2/8/2011

consider $(-,-,+)$

- task is to create E ebits w/ $|Q|$ qubits + $|C|$ cbits.
- points are of the form (C, Q, E) where $C \leq 0$, $Q \leq 0$, + $E \geq 0$

suffices to prove $E \leq |Q|$ b/c combining w/ $Q \leq 0$ + $C \leq 0$ gives

$E \leq |C| + |Q|$ (redundant)

$E \leq |Q|$

$0 \leq |C| + 2|Q|$ (trivial)

Suppose protocol $(-|C|, -|Q|, E)$.
combine w/ tele $(-2E, E, -E)$
to get

$$(-|C|, -|Q|, E) + (-2E, E, -E) = (-|C| - 2E, -|Q| + E, 0)$$

the resulting q. comm. had better not be positive b/c class. comm. alone cannot generate it. Then

$$-|Q| + E \leq 0 \Rightarrow E \leq |Q|$$