Lecture 10

2/3/2011

three noiseless protocols

- entanglement distribution
- super-dense coding
- teleportation

- resource counting
- proofs of optimality

Nonlocal unit resources

1) noiseless qubit channel

$$|i\rangle^A \rightarrow |i\rangle^B$$

preserves superpositions so that

$$\alpha|0\rangle^A + \beta|1\rangle^A \rightarrow \alpha|0\rangle^B + |1\rangle^B$$

indicate this resource w/ notation:

$$[q \rightarrow q]$$

2) Noiseless classical bit channel

$$|i\rangle\langle i|^A \rightarrow |i\rangle\langle i|^B$$

$\&$ $\quad |i\rangle\langle j| \rightarrow 0 \qquad$ for $i \neq j$

resource is

$$[c \rightarrow c]$$

2/3/2011

completely dephasing channel is example

$$\Delta(\rho) \to \frac{1}{2}\rho + \frac{1}{2} Z \rho Z$$

$$= |0\rangle\langle 0|\rho|0\rangle\langle 0| + |1\rangle\langle 1|\rho|1\rangle\langle 1|$$

$$= \langle 0|\rho|0\rangle \; |0\rangle\langle 0| + \langle 1|\rho|1\rangle \; |1\rangle\langle 1|$$

preserves a classical mixture

$$\frac{1}{2}\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right)$$

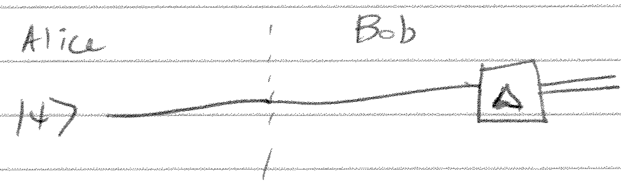
but destroys a superposition state

$$|+\rangle\langle +| = \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)\left(\frac{\langle 0| + \langle 1|}{\sqrt{2}}\right) = \frac{1}{2}\left(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1|\right)$$

$$\longrightarrow \frac{1}{2}\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right)$$

similar to what measurement does

noiseless qubit channel can simulate noiseless classical:



Thus, $[q \to q] \geq [c \to c]$

2/3/2011 (3)

Ebit: shared nonlocal entanglement

$$\frac{1}{\sqrt{2}}\left(|0\rangle^A|0\rangle^B + |1\rangle^A|1\rangle^B\right)$$

Classical comm. alone cannot simulate entanglement or quantum comm.

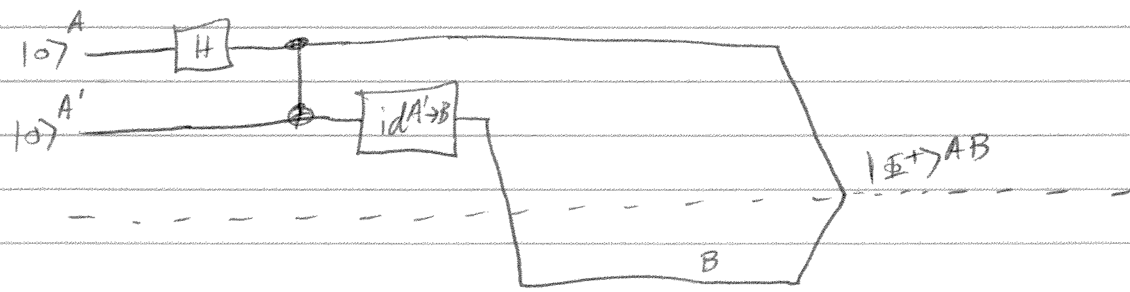
Entanglement alone cannot simulate classical or quantum comm.

1st Protocol: Entanglement Distribution

Two steps

1) Create Bell state locally
2) send half of it over a noiseless qubit channel to create ebit

Diagram



$$[q\to q] \to [qq]$$

1) becomes ebit
2) only count nonlocal resource
3) ~~operations are perfect~~

2/3/2011

Could we have

$\{qq\} \geq \{q \to q\}$?

no, violates no-signaling

$\therefore$ quantum comm. is a stronger comm. resource than entanglement

---

2nd protocol: Super-dense coding

But first, elementary coding

1) Alice prepares $|0\rangle$ or $|1\rangle$
2) transmits over noiseless qubit channel
3) Bob measures Z

$\therefore \quad \{q \to q\} \geq \{c \to c\}$

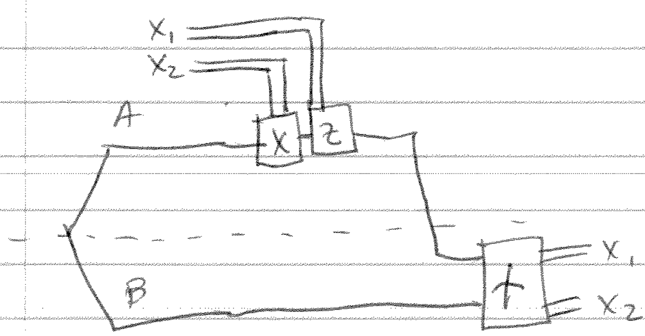
cannot do better than this (homework exercise + more generally Holevo bound) dense coding

1) Alice + Bob share an ebit $|\Phi^+\rangle^{AB}$. Alice applies a Pauli operator to her half of the ebit (one of $\{I, X, Z, XZ\}$). state becomes one of

$$|\Phi^+\rangle^{AB},\ |\Phi^-\rangle^{AB},\ |\Psi^+\rangle^{AB},\ |\Psi^-\rangle^{AB}$$

2) transmits A over noiseless qubit channel

3) Bob performs Bell measurement to distinguish these states



resource inequality is

$$[qq] + [q \to q] \geq 2[c \to c]$$

Note: could have done

Also: protocol is private

$$2[q \to q] \geq 2[c \to c]$$

but dense coding only requires weaker resource of entanglement

2/3/2011

3rd protocol : teleportation

Alice + Bob "swap their equipment"

Alice + Bob share ebit $|\Phi^+\rangle^{AB}$

Alice wants to transmit qubit

$$|\psi\rangle^{A'} \equiv \alpha|0\rangle^{A'} + \beta|1\rangle^{A'}$$

overall state is

$$|\psi\rangle^{A'}|\Phi^+\rangle^{AB}$$

$$= \left(\alpha|0\rangle^{A'} + \beta|1\rangle^{A'}\right)\left(\frac{|00\rangle^{AB} + |11\rangle^{AB}}{\sqrt{2}}\right)$$

$$= \frac{1}{\sqrt{2}}\Big(\alpha|000\rangle^{A'AB} + \alpha|011\rangle^{A'AB} + \beta|100\rangle^{A'AB} + \beta|111\rangle^{A'AB}\Big)$$

use results from homework to write as

$$= \frac{1}{2}\Big\{\alpha\left(|\Phi^+\rangle^{A'A} + |\Phi^-\rangle^{A'A}\right)|0\rangle^B + \beta\left(|\Psi^+\rangle^{A'A} - |\Psi^-\rangle^{A'A}\right)|0\rangle^B + \alpha\left(|\Psi^+\rangle^{A'A} + |\Psi^-\rangle^{A'A}\right)|1\rangle^B + \beta\left(|\Phi^+\rangle^{A'A} - |\Phi^-\rangle^{A'A}\right)|1\rangle^B\Big\}$$

$$= \frac{1}{2}\Big\{ |\Phi^+\rangle^{A'A}\left(\alpha|0\rangle^B + \beta|1\rangle^B\right) +$$
$$|\Phi^-\rangle^{A'A}\left(\alpha|0\rangle^B - \beta|1\rangle^B\right) +$$
$$|\Psi^+\rangle^{A'A}\left(\alpha|1\rangle^B + \beta|0\rangle^B\right) +$$
$$|\Psi^-\rangle^{A'A}\left(\alpha|1\rangle^B - \beta|0\rangle^B\right)\Big\}$$

$$= \frac{1}{2}\Big[|\Phi^+\rangle^{A'A}|\psi\rangle^B + |\Phi^-\rangle^{A'A} Z|\psi\rangle^B +$$
$$|\Psi^+\rangle^{A'A} X|\psi\rangle^B + |\Psi^-\rangle^{A'A} XZ|\psi\rangle^B\Big]$$

Now can outline protocol

1) Alice performs Bell measurement on her systems $A'A$. State collapses to

$$|\Phi^+\rangle^{A'A}\,|\psi\rangle^B,$$
$$|\Phi^-\rangle^{A'A}\,Z|\psi\rangle^B,$$
$$|\Psi^+\rangle^{A'A}\,X|\psi\rangle^B,$$
$$|\Psi^-\rangle^{A'A}\,XZ|\psi\rangle^B$$

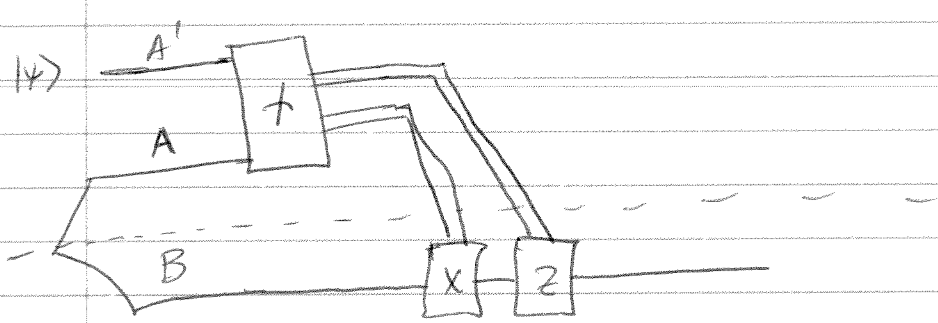
w/ <u>uniform</u> probability $1/4$

Alice knows which state Bob has.

Bob knows nothing. His state ~~~~ is π^B.

2) Alice transmits two classical bits to Bob. Bob then knows exactly which state he has

3) Bob performs a restoration operation



Note: 1) teleportation process works for any input state (universal)

2) does not violate no-cloning theorem (state is teleported, not copied)

3) not instantaneous

resource inequality is

$$[qq] + 2[c \to c] \geq [q \to q]$$

2/3/2011

Optimality of the Three Protocols

ED : $\{q \to q\} \geq \{qq\}$

SD : $\{q \to q\} + \{qq\} \geq 2\{c \to c\}$

TP : $2\{c \to c\} + \{qq\} \geq \{q \to q\}$

are these rates optimal?

1^{st}: could there be some protocol where

$$\{q \to q\} \geq E\{qq\}$$

& $E > 1$?

Suppose it exists.

Assuming free classical comm., TP becomes $\{qq\} \geq \{q \to q\}$ can combine w/ TP to get

$$\{q \to q\} \geq E\{qq\} \geq E\{q \to q\}$$

can then bootstrap & achieve an unbounded amount of quantum communication, which is impossible (later justify w/ quantum capacity theorem

2/3/2011

What about optimality of dense coding?

Suppose unlimited amount of entanglement + there is some protocol that gives ~~suppose dense coding generates~~

$$\{q \to q\} + \infty\{qq\} \geq 2R\{c \to c\} + \infty\{qq\}$$

where $R > 1$

then can chain this w/ TP:

$$\{q \to q\} + \infty\{qq\} \geq 2R\{c \to c\} + \infty\{qq\}$$

$$\geq R\{q \to q\} + \infty\{qq\}$$

can chain over + over + achieve

$$\{q \to q\} + \infty\{qq\} \geq R^k\{q \to q\} + \infty\{qq\}$$

+ make arbitrarily high amount of quantum comm. from just one noiseless qubit + entanglement

violates no-signaling

(can later justify w/ EA capacity theorem)

2/3/2011

Three Noiseless Qudit Protocols

- require this form for use w/ protocols of arbitrary dimension

noiseless qudit channel:

$|i\rangle^A \to |i\rangle^B$ $\log_2 d$ qubits

noiseless cdit channel:

$|i\rangle\langle i|^A \to |i\rangle\langle i|^B$ $\log_2 d$ cdits

$|i\rangle\langle j|^A \to 0$ $i \neq j$

noiseless edit is

$$|\Phi\rangle^{AB} = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |i\rangle^A |i\rangle^B$$

$\log_2 d$ ebits

Entanglement Distribution

Alice does $CNOT_d \cdot F^A |0\rangle^A |0\rangle^{A'}$

where $F : |l\rangle \to \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} \exp\left\{\frac{2\pi i l j}{d}\right\} |j\rangle$

$$CNOT_d = \sum_{j=0}^{d-1} |j\rangle\langle j|^A \otimes X(j)^{A'}$$

2/3/2011

so that

$$F^A |0\rangle^A |0\rangle^{A'} = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} |j\rangle^A |0\rangle^{A'}$$

$$\xrightarrow{\text{CNOT}} \sum_{j'=0}^{d-1} \left(|j'\rangle\langle j'| \otimes X(j') \right) \left(\frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} |j\rangle^A |0\rangle^{A'} \right)$$

$$= \frac{1}{\sqrt{d}} \sum_{j',j=0}^{d-1} |j'\rangle^A \langle j'|j\rangle \otimes |j'\rangle^{A'}$$

$$= \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} |j\rangle^A |j\rangle^{A'} = |\Phi\rangle^{AA'}$$

$$\xrightarrow{\text{noiseless qudit channel}} |\Phi\rangle^{AB}$$

resource inequality: $\log d \{q \to q\} \geq \log d \{qq\}$

Qudit Dense Coding

1) Alice + Bob share

$$|\Phi\rangle^{AB}$$

Alice applies one of $\{X(x) Z(z)\}_{x,z=0}^{d-1}$

where $X(x)|j\rangle = |j \oplus x\rangle$

$$Z(z)|j\rangle = \exp\left\{\frac{i 2\pi z j}{d}\right\} |j\rangle$$

2) Alice sends her half of state $|\Phi_{x,z}\rangle^{AB}$ over qudit channel.

2/3/2011

3) Bob measures in basis

$$\{|\Phi_{x,z}\rangle^{AB}\}$$ (basis is orthonormal,

resource inequality is

$$\log d\,\{q\to q\} + \log d\,\{qq\} \geq 2\log d\,\{c\to c\}$$

Qudit Teleportation

global state is

$$|\psi\rangle^{A'}|\Phi\rangle^{AB}$$

where $|\psi\rangle^{A'} = \sum_{i=0}^{d-1} \alpha_i |i\rangle^{A'}$

& $|\Phi\rangle^{AB} = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} |j\rangle^{A}|j\rangle^{B}$

1) Alice measures $A'A$ in Bell basis

$$\{|\Phi_{i,j}\rangle^{A'A}\}$$

2) transmits i & j to Bob

3) Bob does $Z^B(-j)\,X^B(-i)$

2/3/2011

Analyze post-measurement state (unnormalized

$$\underbrace{|\Phi_{i,j}\rangle\langle\Phi_{i,j}|}_{\text{projector for } |\Phi_{i,j}\rangle}^{A'A} \quad |\psi\rangle^{RA'} \quad |\Phi\rangle^{AB}$$

more general w/ reference system

$$|\Phi_{i,j}\rangle^{A'A} \langle\Phi|^{A'A} \left(U_{i,j}^{A'}\right)^{\dagger} |\psi\rangle^{RA'} |\Phi\rangle^{AB}$$

↓ transpose trick

$$|\Phi_{i,j}\rangle^{A'A} \langle\Phi|^{A'A} \left(U_{i,j}^{T\,A}\right)^{\dagger} |\psi\rangle^{RA'} |\Phi\rangle^{AB}$$

commute

$$|\Phi_{i,j}\rangle^{A'A} \langle\Phi|^{A'A} |\psi\rangle^{RA'} \; U_{i,j}^{*\,A} \; |\Phi\rangle^{AB}$$

↓ transpose trick

$$|\Phi_{i,j}\rangle^{A'A} \langle\Phi|^{A'A} |\psi\rangle^{RA'} \left(U_{i,j}^{\dagger}\right)^{B} |\Phi\rangle^{AB}$$

commute

$$(*) \quad \left(U_{i,j}^{\dagger}\right)^{B} |\Phi_{i,j}\rangle^{A'A} \underbrace{\langle\Phi|^{A'A} |\psi\rangle^{RA'} |\Phi\rangle^{AB}}_{\text{evaluate}}$$

2/3/2011

$$\langle\Phi|^{A'A}\ |\psi\rangle^{RA'}\ |\Phi\rangle^{AB}$$

$$= \langle\Phi|^{A'A}\ |\Phi\rangle^{AB}\ |\psi\rangle^{RA'}$$

$$= \left(\frac{1}{\sqrt{d}}\sum_{i=0}^{d-1}\langle i|^{A'}\langle i|^{A}\right)\left(\frac{1}{\sqrt{d}}\sum_{j=0}^{d-1}|j\rangle^{A}|j\rangle^{B}\right)|\psi\rangle^{RA'}$$

$$= \left(\frac{1}{d}\sum_{i,j=0}^{d-1}\langle i|^{A'}\langle i|j\rangle^{A}|j\rangle^{B}\right)|\psi\rangle^{RA'}$$

$$= \left[\frac{1}{d}\sum_{i=0}^{d-1}\langle i|^{A'}|i\rangle^{B}\right]|\psi\rangle^{RA'}$$

$$= \frac{1}{d}\underbrace{\left[\sum_{i=0}^{d-1}|i\rangle^{B}\langle i|^{A'}\right]}_{\text{noiseless qudit channel from } A' \text{ to } B}|\psi\rangle^{RA'}$$

$$= \frac{1}{d}I^{A'\to B}\,|\psi\rangle^{RA'}$$

$$= \frac{1}{d}|\psi\rangle^{RB}$$

2/3/2011

Back to (*)

$$\frac{1}{d}\left(U_{ij}^{\dagger}\right)^{B} |\Phi_{ij}\rangle^{A'A} |\psi\rangle^{RB}$$

$$= \frac{1}{d} |\Phi_{ij}\rangle^{A'A} \left(U_{ij}^{\dagger}\right)^{B} |\psi\rangle^{RB} \equiv |\phi\rangle^{A'ARB}$$

$$\mathrm{Tr}\left\{ |\phi\rangle\langle\phi|^{A'ARB} \right\} = \frac{1}{d^2}$$

probability uniform & independent of state $|\psi\rangle^{RB}$

Bob's state w/o classical data is

$$\frac{1}{d^2} \sum_{i,j} U_{ij}^{B} |\psi\rangle\langle\psi|^{RB} \left(U_{ij}^{\dagger}\right)^{B} =$$

$$\psi^{R} \otimes \pi^{B}$$

w/ classical data, it is

$\left(U_{ij}^{\dagger}\right)^{B} |\psi\rangle^{RB}$ & he applies U_{ij} to bring it to $|\psi\rangle^{RB}$

teleportation!