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Lecture 9

Example: Erasure channel

Recall: $\rho \rightarrow (1-\epsilon)\rho + \epsilon |e\rangle\langle e|$

Claim: Kraus operators are

$$\{ \sqrt{1-\epsilon}\ |0\rangle\langle 0| + |1\rangle\langle 1|,\ \sqrt{\epsilon}\ |e\rangle\langle 0|,\ \sqrt{\epsilon}\ |e\rangle\langle 1| \}$$

Proof:

$$\sqrt{1-\epsilon}\ (|0\rangle\langle 0| + |1\rangle\langle 1|)\ \rho\ (\sqrt{1-\epsilon}\ \{|0\rangle\langle 0| + |1\rangle\langle 1|\})$$

$$+ \sqrt{\epsilon}\ |e\rangle\langle 0|\ \rho\ |0\rangle\langle e| \sqrt{\epsilon}$$

$$+ \sqrt{\epsilon}\ |e\rangle\langle 1|\ \rho\ |1\rangle\langle e| \sqrt{\epsilon}$$

$$= (1-\epsilon)\rho + \epsilon\ |e\rangle\langle e|\ \langle 0|\rho|0\rangle$$

$$+ \epsilon\ |e\rangle\langle e|\ \langle 1|\rho|1\rangle$$

$$= (1-\epsilon)\rho + \epsilon\ |e\rangle\langle e|$$

Claim: **Isometric Extension**

$$U^{A\to BE} = \sqrt{1-\epsilon}\ |0\rangle^B\langle 0|^A + |1\rangle^B\langle 1|^A \otimes |e\rangle^E +$$

$$\sqrt{\epsilon}\ |e\rangle^B\langle 0|^A \otimes |0\rangle^E +$$

$$\sqrt{\epsilon}\ |e\rangle^B\langle 1|^A \otimes |1\rangle^E$$

Proof

$\mathrm{Tr}_E\{U^{A\to BE}\,\rho\,(U^{A\to BE})^\dagger\}$

$= \sum_i A_i \rho A_i^\dagger$ where A_i are as before

can rewrite isometric extension of erasure channel suggestively

$$U^{A\to BE} = \sqrt{1-\epsilon}\left(I^{A\to B} \otimes |e\rangle^E\right) + \sqrt{\epsilon}\left(|0\rangle^E\langle 0|^A \otimes |e\rangle^B + |1\rangle^E\langle 1|^A \otimes |e\rangle^B\right)$$

$$= \sqrt{1-\epsilon}\left(I^{A\to B} \otimes |e\rangle^E\right) + \sqrt{\epsilon}\left(I^{A\to E} \otimes |e\rangle^B\right)$$

What happens if we trace over Bob?

get erasure channel to Eve w/ opposite probabilities

$\rho \to \epsilon \rho^E + (1-\epsilon)|e\rangle\langle e|^E$

Notice: At $\epsilon = 1/2$, channel to Bob is

$$\rho \to \frac{1}{2}\rho^B + \frac{1}{2}|e\rangle\langle e|^B$$

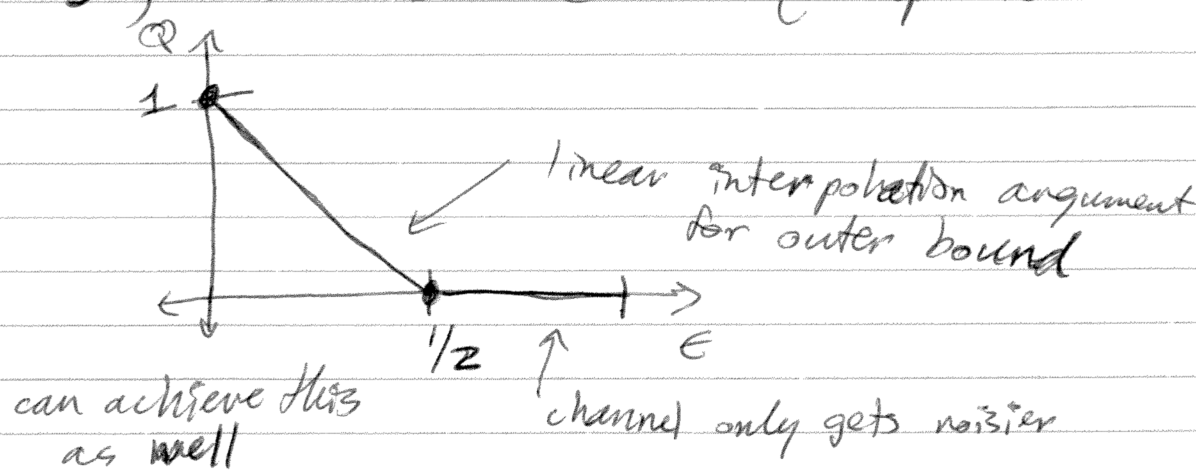
+ channel to Eve is

$$\rho \to \frac{1}{2}\rho^E + \frac{1}{2}|e\rangle\langle e|^E$$

same channel!

∴ If Bob could recover quantum information perfectly from his channel w/ some coding strategy, then Eve could as well by applying the same strategy, + they both would get copies of the input state.
∴ the quantum capacity should be zero when $\epsilon = 1/2$

So, outer bound on q. cap is



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In general, complementary channel to Eve obtained by tracing out Bob.

Suppose channel $\mathcal{N}(\rho) = \sum_j A_j \rho A_j^\dagger$

"canonical" isometric extension is

$$U^{A\to BE} \equiv \sum_j A_j \otimes |j\rangle^E$$

Then $\mathrm{Tr}_B\left\{ U^{A\to BE} \rho \left(U^{A\to BE}\right)^\dagger \right\}$

$$= \mathrm{Tr}_B\left\{ \sum_j A_j \otimes |j\rangle^E \rho \sum_i A_i^\dagger \otimes \langle i|^E \right\}$$

$$= \mathrm{Tr}_B\left\{ \sum_{j,i} A_j \rho A_i^\dagger \otimes |j\rangle\langle i|^E \right\}$$

$$= \sum_{j,i} \mathrm{Tr}\left\{ A_j \rho A_i^\dagger \right\} |j\rangle\langle i|$$

Eve gets density operator w/ matrix elements

$$\mathrm{Tr}\left\{ A_j \rho A_i^\dagger \right\}$$

equivalent up to change of basis on Eve's system

(doesn't really matter b/c it doesn't change entropies for Eve)

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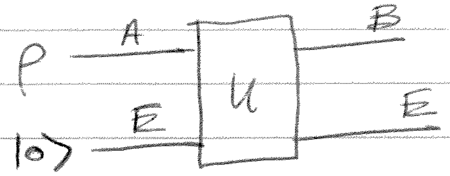
Another equivalent viewpoint:



channel evolution given by

$$\mathcal{N}(\rho) = \mathrm{Tr}_E\left\{ U \left(\rho \otimes |0\rangle\langle 0|^E \right) U^\dagger \right\}$$

where U is a unitary and acts on $A \otimes E$ space

can find Kraus operators

$$\mathcal{N}(\rho) = \sum_i \langle i|^E \left(U \left(\rho^A \otimes |0\rangle\langle 0|^E \right) U^\dagger \right) |i\rangle^E$$

$$= \sum_i \langle i|^E U |0\rangle^E \; \rho \; \langle 0|^E U^\dagger |i\rangle^E$$

Kraus operators are then $\left\{ \langle i| U |0\rangle^E \right\}$

this is an operator

Do they satisfy $\sum_i A_i^\dagger A_i$

check: $\sum_i \langle 0|^E U^\dagger |i\rangle^E \langle i|^E U |0\rangle^E$

$$= \langle 0|^E U^\dagger \left(\sum_i |i\rangle\langle i|^E \right) U |0\rangle^E$$

$$= \langle 0|^E U^\dagger U |0\rangle^E$$

$$= \langle 0|^E I^A \otimes I^E |0\rangle^E$$

$$= I^A$$

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Another example of a channel

Generalized dephasing channel

Extension Acts as $U^{A\to BE}|x\rangle^A = |x\rangle^B|\varphi_x\rangle^E$

where $|\varphi_x\rangle^E$ is some arbitrary basis

can write U as

$$U^{A\to BE} = \sum_x |x\rangle^B|\varphi_x\rangle^E\langle x|^A$$

+ action on density operator is

$$U\rho U^\dagger$$
$$= \left(\sum_x |x\rangle^B|\varphi_x\rangle^E\langle x|^A\right)\rho\left(\sum_{x'}|x'\rangle\langle x'|^B\langle\varphi_{x'}|^E\right)$$
$$= \sum_{x,x'}\langle x|^A\rho|x'\rangle^A\ |x\rangle\langle x'|^B\ |\varphi_x\rangle\langle\varphi_{x'}|^E$$

Trace out E to get action of channel

$$N(\rho) = \sum_{x,x'}\langle x|\rho|x'\rangle\langle\varphi_{x'}|\varphi_x\rangle|x\rangle\langle x'|^B$$

What is channel doing?

nothing to diagonal elements of ρ,
but multiplying off-diagonal terms by arbitrary phases

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Trace out Bob to get complementary map

$$\mathcal{N}^c(\rho) = \underbrace{\sum_x \langle x|\rho|x\rangle |\varphi_x\rangle\langle\varphi_x|}_{\text{this is cq channel}}$$

environment gets no quantum coherence w/ this channel.

More generally, we can define a quantum Hadamard channel as one where the map to the environment is a cq or entanglement-breaking channel