1/27/2011

Lecture 8

Other examples of Noisy Channels

Pauli channel

$$\rho \rightarrow \sum_{i,j=0}^{1} p(i,j)\, Z^i X^j \rho X^j Z^i$$

OR $\rightarrow p_I \rho + p_X X\rho X + p_Y Y\rho Y + p_Z Z\rho Z$

Bloch vector

$\frac{1}{2}(I + r_x X + r_y Y + r_z Z) \rightarrow$

$$\frac{1}{2}\Big(I + (p_I + p_X - p_Y - p_Z) r_x X + (p_I + p_Y - p_X - p_Z) r_Y Y + (p_I + p_Z - p_X - p_Y) r_Z Z\Big)$$

Depolarizing Channel

"Worst-case scenario" channel

$$\rho \rightarrow (1-p)\rho + p\pi$$

can rewrite as

$$(1-p)\rho + p\left(\frac{1}{4}\rho + \frac{1}{4}X\rho X + \frac{1}{4}Y\rho Y + \frac{1}{4}Z\rho Z\right)$$

$$= \left(1 - \frac{3}{4}p\right)\rho + \frac{p}{4}X\rho X + \frac{p}{4}Y\rho Y + \frac{p}{4}Z\rho Z$$

1/27/2011

Action on Bloch vector

$$\frac{1}{2}(I + r_x X + r_y Y + r_z Z) \to \frac{1}{2}\big(I + r_x(1-p)X + r_y(1-p)Y + r_z(1-p)Z\big)$$

as $p \to 1$, Bloch vector shrinks to maximally mixed state $(0,0,0)$

channel sometimes looks classical, sometimes very quantum + strange despite its simplicity

---

Amplitude damping channel (downward decay only)

think of $|0\rangle$ as ground state + $|1\rangle$ as excited state

Spontaneous emission is a process that decays $|1\rangle$ to $|0\rangle$ at some rate γ

- an operator that captures this behavior is $A_0 = \sqrt{\gamma}\,|0\rangle\langle 1|$

so that

$$A_0 |1\rangle\langle 1| A_0^\dagger = \gamma |0\rangle\langle 0|$$

$$+ \ A_0 |0\rangle\langle 0| A_0^\dagger = 0$$

- not a physical map on its own

require some other operator A_1 so that $A_0^\dagger A_0 + A_1^\dagger A_1 = I$

1/27/2011

$$\Rightarrow \quad A_1^\dagger A_1 = I - A_0^\dagger A_0$$
$$= |0\rangle\langle 0| + |1\rangle\langle 1| - \gamma\, |1\rangle\langle 1|$$
$$= |0\rangle\langle 0| + (1-\gamma)\, |1\rangle\langle 1|$$

can then choose

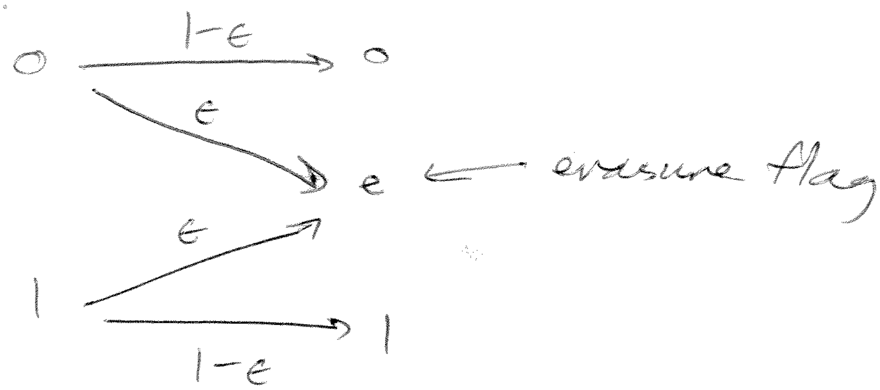
$$A_1 = |0\rangle\langle 0| + \sqrt{1-\gamma}\, |1\rangle\langle 1|$$

quantum capacity known, not much known about classical capacity

---

Erasure Channel

Classical Definition



quantum generalization

$$\underbrace{\rho}_{} \rightarrow \underbrace{(1-\epsilon)\,\rho + \epsilon\, |e\rangle\langle e|}_{}$$

input space $\{|0\rangle, |1\rangle\}$

output space $\{|0\rangle, |1\rangle, |e\rangle\}$

Bob can measure output $\{|0\rangle\langle 0| + |1\rangle\langle 1|,\ |e\rangle\langle e|\}$ to learn whether he receives qubit or erasure symbol

Every capacity for erasure channel is known
(no surprises when used on its own
big surprises when used w/ other channels)

Classical - Quantum Channel
(Measure + Prepare channel)

- channel first measures input & outputs state conditional on outcome of measurement

- just use von Neumann measurement $\{|k\rangle\langle k|\}$

- given outcome k, post-measurement state is

$$\frac{|k\rangle\langle k|\rho|k\rangle\langle k|}{\langle k|\rho|k\rangle}$$

- channel correlates some density operator w/ outcome k:

$$\frac{|k\rangle\langle k|\rho|k\rangle\langle k|}{\langle k|\rho|k\rangle} \otimes \sigma_k$$

ensemble is

$$\left\{\langle k|\rho|k\rangle, \frac{|k\rangle\langle k|\rho|k\rangle\langle k|}{\langle k|\rho|k\rangle} \otimes \sigma_k\right\}$$

dens. op. of ensemble is

$$\sum_k \langle k|\rho|k\rangle \frac{|k\rangle\langle k|\rho|k\rangle\langle k|}{\langle k|\rho|k\rangle} \otimes \sigma_k = \sum_k |k\rangle\langle k|\rho|k\rangle\langle k| \otimes \sigma_k$$

- channel just outputs conditional state

$$\rho \to \sum_k \langle k|\rho|k\rangle \sigma_k$$

- channel is the same as

$$k \to \sigma_k$$ (classical input, quantum output)

Quantum Instrument

- noisy device w/ classical + quantum outputs
- can see as only a partial loss of a measurement outcome.
- suppose there is a measurement w/ two outcomes j + k + measurement operators $\{M_{j,k}\}$

$$p_{J,K}(j,k) = \mathrm{Tr}\{M_{j,k}^\dagger M_{j,k}\,\rho\}$$

$$+ \quad \rho \to \frac{M_{j,k}\,\rho\, M_{j,k}^\dagger}{p_{J,K}(j,k)}$$

marginal probabilities are

$$p_J(j) = \sum_k \mathrm{Tr}\{M_{j,k}^\dagger M_{j,k}\,\rho\}$$

$$p_K(k) = \sum_j \mathrm{Tr}\{M_{j,k}^\dagger M_{j,k}\,\rho\}$$

Suppose measuring device places outcomes in classical registers

$$\Rightarrow \quad \frac{M_{j,k}\, \rho M_{j,k}^{\dagger}}{p_{J,K}(j,k)} \otimes |j\rangle\langle j|^{J} \otimes |k\rangle\langle k|^{K}$$

evolution is then given by

$$\rho \rightarrow \sum_{j,k} M_{j,k}\, \rho M_{j,k}^{\dagger} \otimes |j\rangle\langle j|^{J} \otimes |k\rangle\langle k|^{K}$$

Suppose now we donot have access to K, evolution is then

$$\sum_{j,k} M_{j,k}\, \rho M_{j,k}^{\dagger} \otimes |j\rangle\langle j|^{J}$$

can write as

$$\sum_{j} \mathcal{E}_j(\rho) \otimes |j\rangle\langle j|^{J}$$

where $\mathcal{E}_j(\rho) = \sum_{k} M_{j,k}\, \rho M_{j,k}^{\dagger}$

$\mathcal{E}_j(\rho)$ is completely-positive & trace-reducing

$$\mathrm{Tr}\{\mathcal{E}_j(\rho)\} = p_J(j) \quad \leftarrow \text{implicit dependence on state } \rho$$

quantum output of instrument is $\sum_j \mathcal{E}_j(\rho)$ & $\mathrm{Tr}\{\sum_j \mathcal{E}_j(\rho)\} = 1$

1/27/2011

Purified Quantum Theory

- strange viewpoint - lack of information results from entanglement w/ another system to which we do not have access
  - simple example $|\Phi^+\rangle^{AB}$, but π^A or π^B locally

Purification

Suppose $\rho^A = \sum_x p_X(x)\, |x\rangle\langle x|^A$

purification is some $|\psi\rangle^{RA}$ such that

$$\rho^A = \mathrm{Tr}_R \left\{ |\psi\rangle\langle\psi|^{RA} \right\}$$

simple example of a purification is

$$|\psi\rangle^{RA} = \sum_x \sqrt{p(x)}\, |x\rangle^R |x\rangle^A$$

why?

$$|\psi\rangle\langle\psi|^{RA} = \left(\sum_x \sqrt{p(x)}\, |x\rangle^R |x\rangle^A\right)\left(\sum_{x'} \sqrt{p(x')}\, \langle x'|^R \langle x'|^A\right)$$

$$= \sum_{x,x'} \sqrt{p(x)p(x')}\; |x\rangle\langle x'|^R\, |x\rangle\langle x'|^A$$

$$\mathrm{Tr}_R \{ |\psi\rangle\langle\psi|^{RA} \} =$$

$$\sum_{x,x'} \sqrt{p(x)p(x')} \; \langle x'|x\rangle \; |x\rangle\langle x'|^A$$

$$= \sum_x p(x) |x\rangle\langle x|$$

Observe: All purifications are related by isometries (some operator U such that $U^\dagger U = I$)

Start from Schmidt form & If we have

$$U^R |\psi\rangle^{RA} \text{, then}$$

$$\mathrm{Tr}_R \{ U^R |\psi\rangle\langle\psi|^{RA} (U^\dagger)^R \} = \rho^A$$

(same steps w/

$\langle x'|x\rangle$ replaced w/

$\langle x'|U^\dagger U|x\rangle = \langle x'|x\rangle$

Isometric Extensions

- consider bit-flip channel

(after inputting $|\psi\rangle$) resulting ensemble is

$$\{ \{1-p, |\psi\rangle\}, \{p, X|\psi\rangle\} \}$$

density operator is $(1-p)|\psi\rangle\langle\psi| + p\,X|\psi\rangle\langle\psi|X$

~~purification of this density operator~~ is

$$\sqrt{1-p}\,|\psi\rangle^A|0\rangle^E + \sqrt{p}\,X|\psi\rangle^A|1\rangle^E$$

(check by tracing out E)

- purification system is environment of the channel

can view as a map from A to AE

$$|\psi\rangle^A \longrightarrow \sqrt{1-p}\,|\psi\rangle^A|0\rangle^E + \sqrt{p}\,X|\psi\rangle^A|1\rangle^E$$

can get a matrix representation of this map ~~—~~ two basis states for A & four for AE

$$\begin{array}{c} \\ \langle 0|^A\langle 0|^E \\ \langle 0|^A\langle 1|^E \\ \langle 1|^A\langle 0|^E \\ \langle 1|^A\langle 1|^E \end{array}
\begin{array}{c} \begin{array}{cc} |0\rangle^A & \quad |1\rangle^A \end{array} \\ \begin{bmatrix} \sqrt{1-p} & 0 \\ 0 & \sqrt{p} \\ 0 & \sqrt{1-p} \\ \sqrt{p} & 0 \end{bmatrix} \end{array} = U$$

This is an isometry, check that

$$U^\dagger U = I \quad \text{(on smaller space)}$$

1/27/2011

- Complementary channel is channel to environment (indicates information leaking to the environment - important for quantum capacity of a channel)

- calculate for the bit-flip channel

$$\mathrm{Tr}_A \Big\{ \left(\sqrt{1-p}\, |\psi\rangle^A |0\rangle^E + \sqrt{p}\, X|\psi\rangle^A |1\rangle^E\right) \left(\sqrt{1-p}\, \langle\psi|^A \langle 0|^E + \sqrt{p}\, \langle\psi|^A X\, \langle 1|^E\right) \Big\}$$

$$= \mathrm{Tr}_A \Big\{ (1-p)\, |\psi\rangle\langle\psi|^A\, |0\rangle\langle 0|^E + \sqrt{p(1-p)} \left(X|\psi\rangle\langle\psi|^A\, |1\rangle\langle 0|^E + |\psi\rangle\langle\psi|^A X\, |0\rangle\langle 1|^E \right) + p\, X|\psi\rangle\langle\psi|^A X\, |1\rangle\langle 1|^E \Big\}$$

$$= (1-p)\, |0\rangle\langle 0|^E + \sqrt{p(1-p)}\, \langle\psi|X|\psi\rangle \left(|1\rangle\langle 0|^E + |0\rangle\langle 1|^E\right) + p\, |1\rangle\langle 1|^E$$

$$\langle\psi|X|\psi\rangle = 2\mathrm{RE}\{\alpha^* \beta\}$$

when $p=0$ or 1 environment gets nothing

when $p=1/2$ (worst-case) environment gets what Bob gets (same as what he would get from an X measurement)

channel is $\rho \to \frac{1}{2}\rho + \frac{1}{2} X\rho X$ in such a case

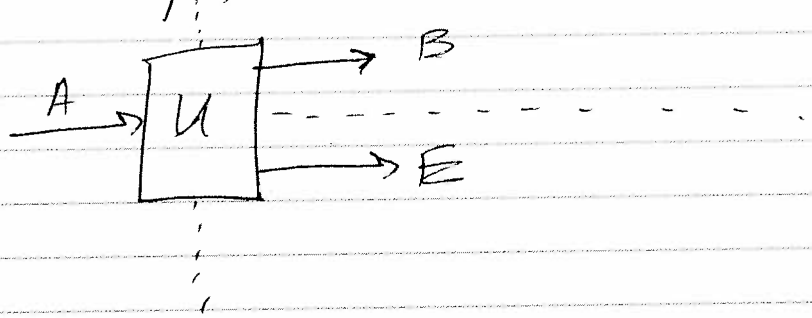
1/27/2011

- can always find an isometric extension for any channel (purification of a channel)

if $\mathcal{N}^{A\to B}(\rho)$,

then $\mathcal{N}^{A\to B}(\rho) = \mathrm{Tr}_E\{ U_{\mathcal{N}}^{A\to BE}\, \rho\, (U_{\mathcal{N}}^{\dagger})^{A\to BE} \}$

pictorially,



isometry $U^{A\to BE}$ satisfies

$$U^{\dagger} U = I^{A}$$

$$U U^{\dagger} = \Pi^{BE}$$

(projector onto some subspace of BE)

Suppose channel

$$\mathcal{N}(\rho) = \sum_j A_j \rho A_j^{\dagger}$$

Then an isometric extension is

$$U_N^{A\to BE} = \sum_j A_j \otimes |j\rangle^E$$

some orthonormal basis for E

"canonical extension"

Why

$$U_N \rho U_N^\dagger = \left(\sum_j A_j \otimes |j\rangle^E\right) \rho \left(\sum_k A_k^\dagger \otimes \langle k|^E\right)$$

$$= \sum_{j,k} A_j \rho A_k^\dagger \otimes |j\rangle\langle k|^E$$

$$\mathrm{Tr}_E\{\cdot\} = \sum_j A_j \rho A_j^\dagger$$

valid isometry?

$$U^\dagger U = \left(\sum_k A_k^\dagger \otimes \langle k|^E\right)\left(\sum_j A_j \otimes |j\rangle^E\right)$$

$$= \sum_{k,j} A_k^\dagger A_j \langle k|j\rangle$$

$$= \sum_j A_j^\dagger A_j = I^A$$

1/27/2011

$$UU^\dagger = \left(\sum_j A_j \otimes |j\rangle^E\right)\left(\sum_k A_k^\dagger \otimes \langle k|^E\right)$$

$$= \sum_{j,k} A_j A_k^\dagger \otimes |j\rangle\langle k|^E$$

Is it true that $UU^\dagger$ is a projector?

I.e., Does $UU^\dagger UU^\dagger = UU^\dagger$

$$\left(\sum_{j,k} A_j A_k^\dagger \otimes |j\rangle\langle k|^E\right)\left(\sum_{j',k'} A_{j'} A_{k'}^\dagger \otimes |j'\rangle\langle k'|^E\right)$$

$$= \sum_{j,k,j',k'} A_j A_k^\dagger A_{j'} A_{k'} \otimes |j\rangle\langle k|j'\rangle\langle k'|$$

$$= \sum_{jkk'} A_j A_k^\dagger A_k A_{k'} \otimes |j\rangle\langle k'|$$

$$= \sum_{j,k'} A_j \left(\sum_k A_k^\dagger A_k\right) A_{k'} \otimes |j\rangle\langle k'|$$

$$= \sum_{j,k'} A_j A_{k'} \otimes |j\rangle\langle k'|$$