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Lecture 7

Composite Noisy Quantum Systems

Suppose two independent ensembles for Alice & Bob

- Alice has $\{p_X(x), |\psi_x\rangle\}$
- Bob has $\{p_Y(y), |\phi_y\rangle\}$

For given x & y, state is $|\psi_x\rangle \otimes |\phi_y\rangle$

density operator of joint system is

$$\mathbb{E}_{X,Y}\left\{ \left(|\psi_X\rangle \otimes |\phi_Y\rangle\right)\left(\langle\psi_X| \otimes \langle\phi_Y|\right) \right\}$$

$$= \mathbb{E}_{X,Y}\left\{ |\psi_X\rangle\langle\psi_X| \otimes |\phi_Y\rangle\langle\phi_Y| \right\}$$

$$= \sum_{x,y} p_X(x)\, p_Y(y)\, |\psi_x\rangle\langle\psi_x| \otimes |\phi_y\rangle\langle\phi_y|$$

$$= \left(\sum_x p_X(x)\, |\psi_x\rangle\langle\psi_x|\right) \otimes \left(\sum_y p_Y(y)\, |\phi_y\rangle\langle\phi_y|\right) \quad \left(\begin{array}{c}\text{TP is} \\ \text{linear}\end{array}\right)$$

$$= \rho \otimes \sigma$$

product state (similar to independent distributions)

$$\xrightarrow{p(x)} X \longrightarrow |\psi_x\rangle$$

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$$\xrightarrow{p(y)} Y \longrightarrow |\phi_y\rangle$$

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Separable States

generate a correlated ensemble

$$\{p_X(x), |\psi_x\rangle \otimes |\phi_x\rangle\}$$

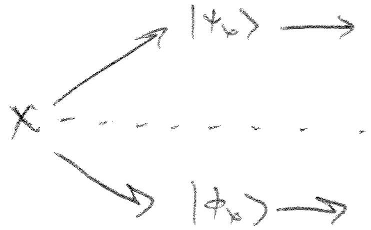
density operator is

$$\sum_x p_X(x)\ |\psi_x\rangle\langle\psi_x| \otimes |\phi_x\rangle\langle\phi_x|$$

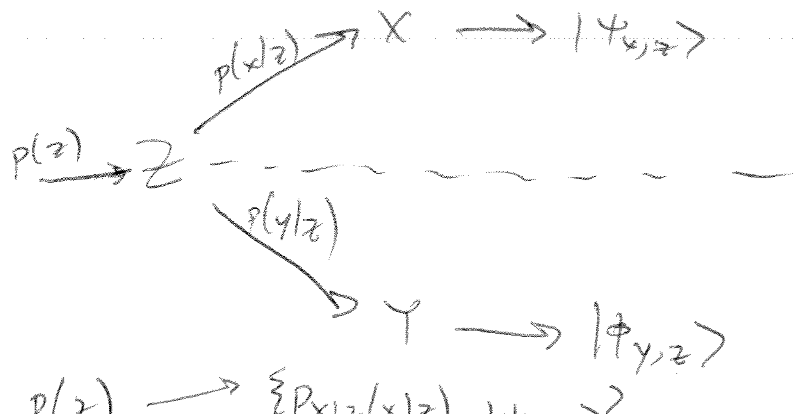
local density operators are

$$\sum_x p_X(x)\, |\psi_x\rangle\langle\psi_x| \qquad \& \qquad \sum_x p_X(x)\, |\phi_x\rangle\langle\phi_x|$$

picture is



can have further mixing



$p_Z(z) \longrightarrow \{p_{X|Z}(x|z), |\psi_{x,z}\rangle\}$

$\searrow \{p_{Y|Z}(y|z), |\phi_{y,z}\rangle\}$

state is

$$\sum_z p_Z(z)\ \rho_z \otimes \sigma_z$$

most general for

Local Density Operator

Suppose global state of Alice & Bob is

$$|\Phi^+\rangle^{AB} = \frac{1}{\sqrt{2}}\left(|00\rangle^{AB} + |11\rangle^{AB}\right)$$

Would like to consider probabilities if Alice performs local measurement w/ measurement operators $\{M_m\}$ where

$$\sum_m M_m = I$$

global measurement operators are $\{M_m^A \otimes I^B\}$
Then probability for outcome m is

$$\mathrm{Tr}\left\{(M_m^A \otimes I^B)|\Phi^+\rangle\langle\Phi^+|\right\} = \langle\Phi^+| M_m^A \otimes I^B |\Phi^+\rangle$$

$$= \frac{1}{2}\sum_{i,j=0}^{1} \langle i|^A \langle i|^B \left(M_m^A \otimes I^B\right)|j\rangle^A|j\rangle^B$$

$$= \frac{1}{2}\sum_{i,j=0}^{1} \langle i|^A M_m^A |j\rangle^A \langle i|^B|j\rangle^B$$

$$= \frac{1}{2}\sum_{i=0}^{1} \langle i| M_m |i\rangle$$

$$= \frac{1}{2}\sum_{i=0}^{1} \mathrm{Tr}\{M_m |i\rangle\langle i|\}$$

$$= \mathrm{Tr}\{M_m \pi\}$$

where $\pi \equiv \frac{1}{2}\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right)$

local measurements give same results as if state is totally random...

local measurement results are the same as for the state

$\pi^A \otimes \pi^B$

$\Rightarrow$ can only distinguish these states w/ global measurements (hwk problem)

Partial Trace

- would like to determine a local density operator that ~~predicts~~ gives probabilities for outcomes of all local measurements
- can use partial trace. suppose state is

$$\mathrm{Tr}_Y\{|x\rangle\langle x|^X \otimes |y\rangle\langle y|^Y\} = |x\rangle\langle x|^X \, \mathrm{Tr}\{|y\rangle\langle y|^Y\}$$
$$= |x\rangle\langle x|^X \langle y|y\rangle$$
$$= |x\rangle\langle x|^X$$

In general, mathematical definition is

$$\mathrm{Tr}_2\{|x_1\rangle\langle x_2| \otimes |y_1\rangle\langle y_2|\} = |x_1\rangle\langle x_2| \, \langle y_1|y_2\rangle$$

+ acts linearly.

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example for arbitrary density operator

ρ^{AB}

expand in terms of some basis $|i\rangle^A \otimes |j\rangle^B$
for joint system

$$\rho^{AB} = \sum_{i,j,k,l} \lambda_{ij,kl} \left(|i\rangle^A \otimes |j\rangle^B \right) \left(\langle k|^A \otimes \langle l|^B \right)$$

$$= \sum_{ijkl} \lambda_{ijkl} \; |i\rangle\langle k|^A \otimes |j\rangle\langle l|^B$$

can now do partial trace

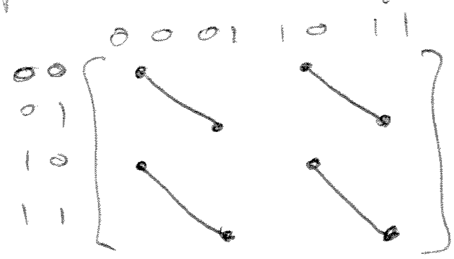
$$\rho^A = Tr_B \left\{ \sum_{ijkl} \lambda_{ijkl} \; |i\rangle\langle k|^A \otimes |j\rangle\langle l|^B \right\}$$

$$= \sum_{ijkl} \lambda_{ijkl} \; |i\rangle\langle k|^A \, \langle l|j\rangle$$

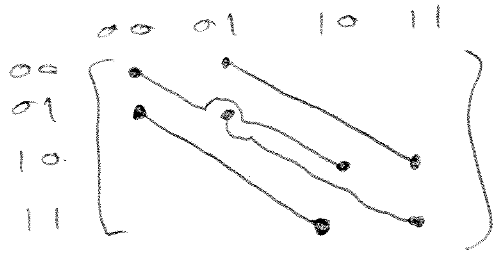
$$= \sum_{ijk} \lambda_{ijkj} \; |i\rangle\langle k|$$

$$= \sum_{i,k} \left(\sum_j \lambda_{ijkj} \right) |i\rangle\langle k|$$

partial trace over 2nd system



partial trace over 1st system



can show that "local" quantum theory is consistent w/ "global" quantum theory

$$\mathrm{Tr}\left\{ \left(M_m^A \otimes I^B \right) \rho^{AB} \right\} = \mathrm{Tr}\left\{ M_m^A \rho^A \right\}$$

(HWK)

can also show that full trace is composed of partial traces

$$\mathrm{Tr}\left\{ M^{AB} \right\} = \mathrm{Tr}_A\left\{ \mathrm{Tr}_B\left\{ M^{AB} \right\}\right\}$$

$$= \mathrm{Tr}_B\left\{ \mathrm{Tr}_A\left\{ M^{AB} \right\}\right\}$$

Classical - Quantum Ensemble

consider ensemble

$$\{ p_X(x), \rho_x \}$$

if it is $\{ p_X(x), |x\rangle\langle x| \}$, then density operator is $\sum_x p_X(x) |x\rangle\langle x|$ + can learn about distribution $p_X(x)$ by performing measurements $\{|x\rangle\langle x|\}$ of the system

(need many "copies" of this ensemble)

In general, ~~ensemble~~ ensemble is $\{p_X(x), \rho_x\}$

& density operator is $\sum_x p_X(x)\rho_x$

difficult to learn about $p_X(x)$ if states ρ_x are mixed.

Alice can solve this problem by correlating a classical state w/ ρ_x so that enlarged ensemble is

$$\{p_X(x), |x\rangle\langle x|^X \otimes \rho_x^A\}$$

classical-quantum ensemble (cq ensemble)
b/c 1st system classical & second system quantum

density operator of enlarged ensemble is

$$\rho^{XA} \equiv \sum_x p_X(x) \, |x\rangle\langle x|^X \otimes \rho_x^A$$

can perform measurement on X to learn about $p_X(x)$

can perform measurement on A to learn about $\rho = \sum_x p(x)\rho_x$

can perform joint measurement to learn about $\{p_X(x), \rho_x\}$

Also, $\rho^{XA} = \bigoplus p_X(x)\rho_x^A = \begin{bmatrix} \boxed{p_X(0)\rho_0} & & \\ & \boxed{p_X(1)\rho_1} & \\ & & \ddots \end{bmatrix}$

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Noisy Evolution

Suppose state is $|\psi\rangle$. send over bit-flip channel.

$$\begin{tikzcd}[row sep=small] & |\psi\rangle \\ |\psi\rangle \arrow[ur, "1-p"] \arrow[dr, "p"'] & \\ & X|\psi\rangle \end{tikzcd}$$

ensemble is $\{\{p, X|\psi\rangle\}, \{1-p, |\psi\rangle\}\}$

density operator is

$$p\, X|\psi\rangle\langle\psi|X + (1-p)|\psi\rangle\langle\psi|$$

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ensemble is $\{p_X(x), |\psi_x\rangle\}$. send through channel

$$\begin{tikzcd}[row sep=small] & & |\psi_x\rangle \\ X \arrow[r, "p_X(x)"] & |\psi_x\rangle \arrow[ur, "1-p"] \arrow[dr, "p"'] & \\ & & X|\psi_x\rangle \end{tikzcd}$$

ensemble is $\{\{p_X(x)p, X|\psi_x\rangle\}, \{p_X(x)(1-p), |\psi_x\rangle\}\}$

density operator is

$$\sum_x p_X(x)\left[p\, X|\psi_x\rangle\langle\psi_x|X + (1-p)|\psi_x\rangle\langle\psi_x|\right]$$

$$= p\, X\rho X + (1-p)\rho$$

state is more mixed than original state (can quantitatively show this later on)

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in general, random unitary channel is

$$\sum_k p(k)\, U_k \rho U_k^\dagger$$

Losing Information from Measurement

$$\{p_X(x), |\psi_x\rangle\}$$

measurement operators $\{M_k\}$ such that $\sum_k M_k^\dagger M_k = I$

Suppose know state. probability of outcome k is

$$p_{K|X}(k|x) = \langle\psi_x| M_k^\dagger M_k |\psi_x\rangle$$

post-measurement state is

$$\frac{M_k|\psi_x\rangle}{\sqrt{p_{K|X}(k|x)}}$$

~~[illegible]~~ suppose we lose track of measurement outcome

ensemble is then

$$\left\{ p_{X|K}(x|k)\, p_K(k),\ \frac{M_k|\psi_x\rangle}{\sqrt{p_{K|X}(k|x)}} \right\}$$

density operator is

$$\sum_{x,k} p_{K|X}(k|x)\, p_X(x) \frac{M_k|\psi_x\rangle\langle\psi_x|M_k^\dagger}{p_{K|X}(k|x)} = \sum_k M_k \rho M_k^\dagger$$
$$= \mathcal{N}(\rho)$$

1/25/2011 evolution is trace-preserving

$$\mathrm{Tr}\left\{\sum_k M_k \rho M_k^\dagger\right\} = \mathrm{Tr}\left\{\sum_k M_k^\dagger M_k \rho\right\}$$
$$= \mathrm{Tr}\{\rho\} = 1$$

Examples of Noisy Channels

Dephasing $\quad \rho \to (1-p)\rho + pZ\rho Z$

action on Bloch Vector is

$$(1-p)\left[\frac{1}{2}\left(I + r_x X + r_y Y + r_z Z\right)\right] +$$
$$p\left\{\frac{1}{2} Z\left(I + r_x X + r_y Y + r_z Z\right) Z\right\}$$
$$= \frac{1}{2}\left\{I + r_x(1-2p)X + r_y(1-2p)Y + r_z Z\right]$$

compresses along X + Y directions while preserving Z direction. for sending classical info., encode in Z direction $\{|0\rangle, |1\rangle\}$

similar behavior for

$$\rho \to (1-p)\rho + pY\rho Y$$
$$+$$
$$\rho \to (1-p)\rho + X\rho X$$

complete dephasing is same as forgetting outcome of Z measurement.