Lecture 6

- can never have perfect knowledge of a state
- errors can also occur in preparation, evolution, or measurement
- relax this assumption & "noisy quantum theory" subsumes probability theory & noiseless quantum theory

Proceed in the following order:

1) density operators
2) general form of measurements
3) composite noisy systems
4) noisy evolution

Noisy States

Suppose a third party prepares a state $|\psi_x\rangle$ w/ prob. $p(x)$, but doesn't tell us which one he prepared

- our best description is as ensemble

$$\mathcal{E} \equiv \{p(x), |\psi_x\rangle\}$$

What is the outcome of a measurement w/ projectors $\{\Pi_j\}$ such that $\sum_j \Pi_j = I$?

Let J denote R.V. for measurement outcome

Suppose that state is $|\psi_x\rangle$.

Then conditional probability for getting outcome j is

$$p_{J|X}(j|x) = \langle \psi_x | \Pi_j | \psi_x \rangle$$

& post-measurement state is

$$\frac{\Pi_j |\psi_x\rangle}{\sqrt{p(j|x)}}$$

But, since we don't know x, the relevant prob. for measurement outcome is <u>unconditional</u> prob. $p_J(j)$

From law of total prob.,

$$p_J(j) = \sum_x p_{J|X}(j|x)\, p_X(x)$$
$$= \sum_x \langle \psi_x | \Pi_j | \psi_x \rangle\, p_X(x)$$

Define the trace of operator A as

$$\mathrm{Tr}\{A\} \equiv \sum_i \langle i | A | i \rangle \quad \text{where } \{|i\rangle\} \text{ o.n. basis}$$

Then

$$\mathrm{Tr}\{\Pi_j |\psi_x\rangle\langle\psi_x|\} = \sum_i \langle i | \Pi_j | \psi_x\rangle \langle \psi_x | i \rangle$$
$$= \sum_i \langle \psi_x | i \rangle \langle i | \Pi_j | \psi_x \rangle$$
$$= \langle \psi_x | \left(\sum_i |i\rangle\langle i| \right) \Pi_j | \psi_x \rangle$$
$$= \langle \psi_x | \Pi_j | \psi_x \rangle$$

Then $$p_J(j) = \sum_x \text{Tr}\{\Pi_j |\psi_x\rangle\langle\psi_x|\} p_X(x)$$

$$= \text{Tr}\left\{\Pi_j \left(\sum_x p_X(x) |\psi_x\rangle\langle\psi_x|\right)\right\}$$

rewrite as

$$p_J(j) = \text{Tr}\{\Pi_j \, \rho\}$$

where ρ is density operator

$$\rho \equiv \sum_x p_X(x) \, |\psi_x\rangle\langle\psi_x|$$

AKA <u>expected</u> density operator for ensemble

$$\rho = \mathbb{E}_X\{|\psi_X\rangle\langle\psi_X|\}$$

<u>Properties of density operator</u>

1) unit trace, 2) positive 3) Hermitian

1) $$\text{Tr}\{\rho\} = \text{Tr}\left\{\sum_x p(x) \, |\psi_x\rangle\langle\psi_x|\right\}$$

$$= \sum_x p(x) \, \text{Tr}\{|\psi_x\rangle\langle\psi_x|\}$$

$$= \sum_x p(x) \, \langle\psi_x|\psi_x\rangle$$

$$= \sum_x p(x) = 1$$

2) $\forall \, |\varphi\rangle$ $\quad \langle\varphi|\rho|\varphi\rangle \geq 0$

$$\langle\varphi|\rho|\varphi\rangle = \langle\varphi|\left(\sum_x p(x) |\psi_x\rangle\langle\psi_x|\right)|\varphi\rangle$$

$$= \sum_x p(x) \, \langle\varphi|\psi_x\rangle\langle\psi_x|\varphi\rangle$$

$$= \sum_x p(x) \, |\langle\psi_x|\varphi\rangle|^2 \geq 0$$

3) $$\rho^\dagger = \left(\sum_x p(x)\,|\psi_x\rangle\langle\psi_x|\right)^\dagger$$
$$= \sum_x p(x)\,\left(|\psi_x\rangle\langle\psi_x|\right)^\dagger$$
$$= \sum_x p(x)\,|\psi_x\rangle\langle\psi_x|$$
$$= \rho$$

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every ensemble has a unique density operator but every density operator does not correspond to a unique ensemble

e.g. $\{\{\frac{1}{2}, |0\rangle\}, \{\frac{1}{2}, |1\rangle\}\}$

$\neq$ $\{\{\frac{1}{2}, |+\rangle\}, \{\frac{1}{2}, |-\rangle\}\}$

have same density operator

$\frac{I}{2}$ (maximally mixed state)

In spite of this, there is a "canonical" ensemble for a given density operator (though still not unique)

since every density ~~operator~~ ρ is Hermitian, can diagonalize it

$$\rho = \sum_{x=0}^{d-1} r_x\,|\phi_x\rangle\langle\phi_x|$$

eigenvals (probabilities) — O.N. eigenvectors

ensemble is then

$\{r_x, |\phi_x\rangle\}$

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can say that the density operator is "the state" b/c we can calculate probabilities w/ it.

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consider pure qubit state

$$|\psi\rangle \equiv \cos\left(\tfrac{\theta}{2}\right)|0\rangle + e^{i\varphi}\sin\left(\tfrac{\theta}{2}\right)|1\rangle$$

density operator is

$$|\psi\rangle\langle\psi| = \left(\cos\left(\tfrac{\theta}{2}\right)|0\rangle + e^{i\varphi}\sin\left(\tfrac{\theta}{2}\right)|1\rangle\right)$$
$$\left(\cos\left(\tfrac{\theta}{2}\right)\langle 0| + e^{-i\varphi}\sin\left(\tfrac{\theta}{2}\right)\langle 1|\right)$$

$$= \cos^2\left(\tfrac{\theta}{2}\right)|0\rangle\langle 0| + e^{i\varphi}\sin\left(\tfrac{\theta}{2}\right)\cos\left(\tfrac{\theta}{2}\right)|1\rangle\langle 0|$$
$$+ e^{-i\varphi}\cos\left(\tfrac{\theta}{2}\right)\sin\left(\tfrac{\theta}{2}\right)|0\rangle\langle 1| + \sin^2\left(\tfrac{\theta}{2}\right)|1\rangle\langle 1|$$

put in matrix "density matrix"

$$\begin{pmatrix} \cos^2\left(\tfrac{\theta}{2}\right) & e^{-i\varphi}\sin\left(\tfrac{\theta}{2}\right)\cos\left(\tfrac{\theta}{2}\right) \\ e^{i\varphi}\sin\left(\tfrac{\theta}{2}\right)\cos\left(\tfrac{\theta}{2}\right) & \sin^2\left(\tfrac{\theta}{2}\right) \end{pmatrix}$$

Trig identities gives

$$\frac{1}{2}\begin{pmatrix} 1+\cos\theta & \sin\theta(\cos\varphi - i\sin\varphi) \\ \sin\theta(\cos\varphi + i\sin\varphi) & 1-\cos\theta \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1+r_z & r_x - i r_y \\ r_x + i r_y & 1-r_z \end{pmatrix}$$

$r_x = \sin\theta\cos\varphi$

$r_y = \sin\theta\sin\varphi$

$r_z = \cos\theta$

can use Pauli matrices I, X, Y, Z
to write as

$$\frac{1}{2}\left(I + r_x X + r_y Y + r_z Z\right)$$

$$\vec{r} \equiv (r_x, r_y, r_z) \quad \text{- Bloch vector}$$

can write any pure state like this

Also, $\sum_i p(i) |\psi_i\rangle\langle\psi_i| \Rightarrow \vec{r} = \sum_i p(i) \vec{r}_i$

gets points inside the Bloch sphere (not just on the boundary)

very useful when reasoning about qubits

Ensemble of Ensembles

$$\mathcal{F} \equiv \{p(x), \rho_x\}$$

ensemble of density operators
"two layers of randomization"

1) $p(x)$
2) could say that each density operator ρ_x arises from ensemble $\{p_{Y|X}(y|x), |\psi_{x,y}\rangle\}$
so that

$$\rho_x = \mathbb{E}_{Y|X}\{|\psi_{x,Y}\rangle\langle\psi_{x,Y}|\}$$
$$= \sum_y p(y|x) |\psi_{x,y}\rangle\langle\psi_{x,y}|$$

density operator for someone who is ignorant of x is

$$\rho \equiv \sum_{x,y} p(y|x)p(x) |\psi_{x,y}\rangle\langle\psi_{x,y}| = \sum_x p(x)\rho_x$$

Noiseless Evolution of an ensemble

Suppose some ensemble $\{p(x), |\psi_x\rangle\}$

suppose we know the state $|\psi_x\rangle$

Then after evolution U, new state is

$$U|\psi_x\rangle$$

can say that we have a ~~new~~ ensemble

$$\{p(x), U|\psi_x\rangle\}$$

density operator for original ensemble is

$$\rho = \sum_x p(x) |\psi_x\rangle\langle\psi_x|$$

density operator for evolved ensemble is

$$\sum_x p(x)\, U|\psi_x\rangle\langle\psi_x|U^\dagger = U\left(\sum_x p(x)|\psi_x\rangle\langle\psi_x|\right)U^\dagger$$

$$= \boxed{U\rho U^\dagger}$$

evolution of the density operator

Noiseless Measurement

have ensemble $\{p(x), |\psi_x\rangle\}$ again

Suppose for now that we know the state is $|\psi_x\rangle$

+ we perform a measurement $\{\Pi_j\}$

probability for getting j is

$$p_{J|X}(j|x) = \langle\psi_x|\Pi_j|\psi_x\rangle$$ + post measurement ~~state~~ is

$$\frac{\Pi_j|\psi_x\rangle}{\sqrt{p_{J|X}(j|x)}}$$

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Suppose that we perform measurement, but we don't know on which state we performed the measurement (though we know measurement result)

ensemble is then

$$\mathcal{E}_j \equiv \left\{ p_{X|J}(x|j), \frac{\Pi_j |\psi_x\rangle}{\sqrt{p_{J|X}(j|x)}} \right\}$$

we know;

density operator of ensemble is

$$\sum_x p(x|j) \frac{\Pi_j |\psi_x\rangle\langle\psi_x| \Pi_j}{p_{J|X}(j|x)}$$

$$= \Pi_j \left(\sum_x \frac{p(x|j)\, |\psi_x\rangle\langle\psi_x|}{p(j|x)} \right) \Pi_j$$

(Note: $p(x|j) = \frac{p(j|x)}{p(j)}$

$$\therefore \quad = \Pi_j \left(\sum_x \frac{p(j|x)p(x)}{p(j|x)\,p(j)} |\psi_x\rangle\langle\psi_x| \right) \Pi_j$$

$$= \Pi_j \frac{\left(\sum_x p(x) |\psi_x\rangle\langle\psi_x|\right)}{p(j)} \Pi_j$$

$$= \boxed{\frac{\Pi_j \rho \Pi_j}{p(j)}}$$

this is how the density operator evolves under measurement

Use law of total probability to get $p(j)$

$$p_J(j) = \sum_x p_{J|X}(j|x)\, p(x)$$

$$= \sum_x \langle\psi_x|\Pi_j|\psi_x\rangle\, p(x)$$

$$= \sum_x \mathrm{Tr}\{|\psi_x\rangle\langle\psi_x|\Pi_j\}\, p(x)$$

$$= \mathrm{Tr}\left\{\left(\sum_x p(x) |\psi_x\rangle\langle\psi_x|\right)\Pi_j\right\}$$

$$= \mathrm{Tr}\{\rho \Pi_j\}$$

measures the "shadow" of ρ on Π_j subspace

1/20/2011 Probability Theory as a special case of the noisy quantum theory

pick ensemble $\{p(x), |x\rangle\}$

of classical states, meaning $|x\rangle$ are O.N. & thus distinguishable

(can distinguish them w/ a measurement w/ projection operators $\{|x\rangle\langle x|\}$)

analogy of prob. distribution is density operator:

$$p_X(x) \longleftrightarrow \rho$$

Analogy of R.V. X is observable

$$X \longleftrightarrow X = \sum_x x|x\rangle\langle x|$$

$$\mathbb{E}\{X\} = \mathrm{Tr}\{X\rho\}$$

why?

$$\mathrm{Tr}\{X\rho\} = \mathrm{Tr}\{\sum_x x|x\rangle\langle x| \sum_{x'} p(x')|x'\rangle\langle x'|\}$$

$$= \sum_{x,x'} x\, p(x') \langle x|x'\rangle\langle x'|x\rangle$$

$$= \sum_{x,x'} x\, p(x')\, \delta_{x,x'}$$

$$= \sum_x x\, p(x)$$

same formula as classical expectation

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indicator r.v.

$$I_A(x) \equiv \begin{cases} 1 & : x \in A \\ 0 & : x \notin A \end{cases}$$

$$\mathbb{E}\{I_A(X)\} = \sum_{x \in A} p_X(x) \equiv p_X(A)$$

can form an indicator observable

$$I_A(X) = \sum_{x \in A} |x\rangle\langle x|$$

same calculation shows that

$$\mathrm{Tr}\{I_A(X)\,\rho\} = p_X(A)$$

since indicator observable is a projector, we can make a measurement out of it:

$$\{I_A(X), I_{A^c}(X)\}$$

where $I_{A^c}(X) = I - I_A(X)$

$$= \sum_{x \in A^c} |x\rangle\langle x|$$

result of such a measurement is to project onto subspace $I_A(X)$ w/ prob. $p_X(A)$ + onto subspace $I_{A^c}(X)$ w/ prob. $p_X(A^c)$

Further connections:

two disjoint sets $A + B$

probability of union is

$$p(A \cup B) = p(A) + p(B)$$

can do similar calculation in QM

$$\Pi_A = \sum_{x \in A} |x\rangle\langle x|$$

$$\Pi_B = \sum_{x \in B} |x\rangle\langle x|$$

$$\Pi_{A \cup B} = \sum_{x \in A \cup B} |x\rangle\langle x| = \Pi_A + \Pi_B$$

can show that

$$\mathrm{Tr}\{\Pi_{A \cup B}\, \rho\} = p(A) + p(B)$$

Intersection of sets:

$$p(A \cap B)$$

can multiply these projectors to get intersection projector

$$\Pi_A \Pi_B = \sum_{x \in A} |x\rangle\langle x| \sum_{x' \in B} |x'\rangle\langle x'|$$

$$= \sum_{x \in A,\, x' \in B} |x\rangle\langle x'| \; \langle x | x'\rangle$$

$$= \sum_{x \in A,\, x' \in B} |x\rangle\langle x'| \; \delta_{x,x'}$$

$$= \sum_{x \in A \cap B} |x\rangle\langle x|$$

$$= \Pi_{A \cap B}$$

1/20/2011 intuition helpful,

but it all breaks down w/ a little non-orthogonality!

(play w/ $|0\rangle$ + $|+\rangle$, for example)

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projectors not the most general form of measurement

- in general, can be set of operators $\{M_j\}$ such that

$$\sum_j M_j^\dagger M_j = I$$

for pure states, probabilities from measurement are

$$p(j) = \langle\psi| M_j^\dagger M_j |\psi\rangle$$

+ post-measurement state is

$$\frac{M_j |\psi\rangle}{\sqrt{p(j)}}$$

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for mixed states,

$$p(j) = tr\{M_j^\dagger M_j \rho\}$$

+ post-measurement state is

$$\frac{M_j \rho M_j^\dagger}{p(j)}$$

POVM Formalism

useful if we are just interested in classical outcome of measurement, not post-measurement state

(application in classical communication over a quantum channel)

POVM - (positive operator-valued measure)

some operators $\overbrace{\{\underbrace{\Lambda_j}_{\uparrow}\}}^{\text{POVM}}$

POVM elements

$$\forall_j \quad \Lambda_j \geq 0$$

$$\sum_j \Lambda_j = I$$

("just like" probabilities)

If state is ρ, probability for getting outcome j is

$$\text{Tr}\{\Lambda_j \rho\}$$