Lecture 5

1/18/2011

Matrix Representations of Operators

NOT gate - define to act on computational basis as

$$X|i\rangle = |i \oplus 1\rangle \qquad i \in \{0,1\}$$

How does it act on superposition state?
linearly ...

$$X\left(\alpha|0\rangle + \beta|1\rangle\right) = \alpha\left(X|0\rangle\right) + \beta\left(X|1\rangle\right)$$
$$= \alpha|1\rangle + \beta|0\rangle$$

"bit flip"

want a matrix representation for X

- combine w/ bra $\langle 0|$ $\langle 0|X|0\rangle = \langle 0|1\rangle = 0$

$\langle 0|X|1\rangle = \langle 0|0\rangle = 1$

also, combine w/ $\langle 1|$:

$$\langle 1|X|0\rangle = \langle 1|1\rangle = 1$$
$$\langle 1|X|1\rangle = \langle 1|0\rangle = 0$$

put these in a matrix:

$$\begin{bmatrix} \langle 0|X|0\rangle & \langle 0|X|1\rangle \\ \langle 1|X|0\rangle & \langle 1|X|1\rangle \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

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How does NOT act on a different basis +/-?

$$X|+\rangle = X\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)$$

$$= \frac{|1\rangle + |0\rangle}{\sqrt{2}}$$

$$= |+\rangle$$

$|+\rangle$ is an eigenstate of X w/ eigenvalue +1

$$X|-\rangle = X\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$$

$$= \frac{|1\rangle - |0\rangle}{\sqrt{2}}$$

$$= -\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$$

$$= -|-\rangle$$

$|-\rangle$ is an eigenstate of X w/ eigenvalue -1

Matrix representation of X in +/- basis:

$$\begin{bmatrix} \langle +|X|+\rangle & \langle +|X|-\rangle \\ \langle -|X|+\rangle & \langle -|X|-\rangle \end{bmatrix} = \begin{bmatrix} \langle +|+\rangle & -\langle +|-\rangle \\ \langle -|+\rangle & -\langle -|-\rangle \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$ diagonal

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Let Z be the flip operator for the +/- basis

$$Z|+\rangle \rightarrow |-\rangle$$
$$Z|-\rangle \rightarrow |+\rangle$$

matrix representation of Z in +/- basis:

$$\begin{bmatrix} \langle +|Z|+\rangle & \langle +|Z|-\rangle \\ \langle -|Z|+\rangle & \langle -|Z|-\rangle \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

"phase flip"

$$Z\left(\frac{|+\rangle + |-\rangle}{\sqrt{2}}\right) = \frac{|-\rangle + |+\rangle}{\sqrt{2}}$$

$$Z\left(\frac{|+\rangle - |-\rangle}{\sqrt{2}}\right) = \frac{|-\rangle - |+\rangle}{\sqrt{2}} = -\left(\frac{|+\rangle - |-\rangle}{\sqrt{2}}\right)$$

$$\Rightarrow \quad Z|0\rangle = |0\rangle$$
$$Z|1\rangle = -|1\rangle$$

$\Rightarrow$ matrix representation for Z in comp. basis:

$$\begin{bmatrix} \langle 0|Z|0\rangle & \langle 0|Z|1\rangle \\ \langle 1|Z|0\rangle & \langle 1|Z|1\rangle \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{diagonal}$$

Z + X are dual to one another

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Commutator

$[A,B] = AB - BA$ if $[A,B]=0$ then A & B commu

Anticommutator

$\{A,B\} = AB + BA$ $\{A,B\}=0 \Rightarrow A$ & B anticommu

Pauli Matrices

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

representation is in computational basis

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$ – looks this ~~way~~ in any basis

$$Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

$Y = iXZ$ – combination of bit & phase flip

I, X, Y, Z – Pauli matrices (very important...)

Another important evolution – Hadamard

changes from Z basis to X basis

$|0\rangle \to |+\rangle$
$|1\rangle \to |-\rangle$

$\Rightarrow \quad H = |+\rangle\langle 0| + |-\rangle\langle 1|$

check w/ $H|0\rangle = |+\rangle$ & $H|1\rangle = |-\rangle$

can generalize – suppose one O.N. basis is $\{|\psi_i\rangle\}_{i=0,1}$

& another is $\{|\phi_i\rangle\}_{i\in\{0,1\}}$

then we can go from one basis to another w/

$$|\phi_0\rangle\langle\psi_0| + |\phi_1\rangle\langle\psi_1|$$

matrix rep. for Hadamard is

$$\begin{pmatrix} \langle 0|H|0\rangle & \langle 0|H|1\rangle \\ \langle 1|H|0\rangle & \langle 1|H|1\rangle \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$\Rightarrow \quad HH = I$

$\Rightarrow \quad H: \; |+\rangle \to |0\rangle$
$\qquad\qquad |-\rangle \to |1\rangle$

$\Rightarrow H = |0\rangle\langle +| + |1\rangle\langle -|$

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Measurement

- way to "read out" classical info. from quantum states

- suppose we measure in comp. basis
- "measure the Z observable"

for $\alpha|0\rangle + \beta|1\rangle$, this collapses to

$|0\rangle$ w/ prob. $|\alpha|^2$ &

$|1\rangle$ w/ prob $|\beta|^2$

What if we measure in the +/- basis?

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = \frac{\alpha+\beta}{\sqrt{2}}|+\rangle + \frac{\alpha-\beta}{\sqrt{2}}|-\rangle$$

$\Rightarrow$ collapses to

$|+\rangle$ w/ prob $\frac{|\alpha+\beta|^2}{2}$ +

$|-\rangle$ w/ prob $\frac{|\alpha-\beta|^2}{2}$

$\Rightarrow$ quantum interference of prob. amplitudes

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state $\alpha|0\rangle + \beta|1\rangle$ is different from ensemble $\{\{|\alpha|^2, |0\rangle\}, \{|\beta|^2, |1\rangle\}\}$

1st, consider measuring Z operator of $|\psi\rangle$ + ensemble. gives same results

$|0\rangle$ w/ prob. $|\alpha|^2$

$|1\rangle$ w/ prob. $|\beta|^2$

Now, consider measuring X operator

for state, we get $|+\rangle$ w/ prob. $\frac{|\alpha+\beta|^2}{2}$

+ $|-\rangle$ w/ prob. $\frac{|\alpha-\beta|^2}{2}$

for ensemble, we have

$$|0\rangle = \frac{|+\rangle + |-\rangle}{\sqrt{2}} \quad + \quad |1\rangle = \frac{|+\rangle - |-\rangle}{\sqrt{2}}$$

$$\Rightarrow \begin{tikzcd} & & {|+\rangle} \\ & {|0\rangle} \arrow[ur, "1/2"] \arrow[r, "1/2"] & {|-\rangle} \\ \cdot \arrow[ur, "|\alpha|^2"] \arrow[dr, "|\beta|^2"'] & & {|+\rangle} \\ & {|1\rangle} \arrow[ur, "1/2"] \arrow[r, "1/2"'] & {|-\rangle} \end{tikzcd}$$

$\Rightarrow$ $|+\rangle$ w/ prob. $1/2$

$|-\rangle$ w/ prob. $1/2$

Very different physical consequences!

More formal way for describing measurement

write Z as

$$Z = |0\rangle\langle 0| - |1\rangle\langle 1|$$

$$\Pi_0 \equiv |0\rangle\langle 0| \quad \& \quad \Pi_1 = |1\rangle\langle 1|$$

$$\Rightarrow Z = \Pi_0 - \Pi_1$$

Consider $\langle\psi|\Pi_0|\psi\rangle = \langle\psi|0\rangle\langle 0|\psi\rangle$
$= \alpha^* \alpha = |\alpha|^2$

↑ probability for getting $|0\rangle$

$\langle\psi|\Pi_1|\psi\rangle = |\beta|^2$

↑ probability for getting $|1\rangle$

What is post-measurement state?

given by projection + renormalization

$$|\psi\rangle \rightarrow \frac{\Pi_0|\psi\rangle}{\sqrt{\langle\psi|\Pi_0|\psi\rangle}} \quad \text{w/ prob. } \langle\psi|\Pi_0|\psi\rangle$$

$$\searrow \frac{\Pi_1|\psi\rangle}{\sqrt{\langle\psi|\Pi_1|\psi\rangle}} \quad \text{w/ prob. } \langle\psi|\Pi_1|\psi\rangle$$

can also measure in any O.N. basis this way

$\{|\phi_i\rangle\}_{i\in\{0,1\}}$ $\quad |\psi\rangle \longrightarrow \dfrac{|\phi_i\rangle\langle\phi_i|\psi\rangle}{|\langle\phi_i|\psi\rangle|}$ w/ prob. $|\langle\phi_i|\psi\rangle|^2$

Expectation of measurement result

$\alpha|0\rangle + \beta|1\rangle \rightarrow$ measure Z $\quad$ $|0\rangle$ w/ prob $|\alpha|^2$
$|1\rangle$ " " $|\beta|^2$

$$\mathbb{E}[Z] = (1)|\alpha|^2 + (-1)|\beta|^2$$
$$= |\alpha|^2 - |\beta|^2$$

Another way

$$(1)|\alpha|^2 + (-1)|\beta|^2$$
$$= \langle\psi|\Pi_0|\psi\rangle - \langle\psi|\Pi_1|\psi\rangle$$
$$= \langle\psi|\Pi_0 - \Pi_1|\psi\rangle$$
$$= \langle\psi|Z|\psi\rangle \equiv \langle Z\rangle_\psi$$

Variance of measurement result

$$\text{Var}\{Z\} \equiv \mathbb{E}\{Z^2\} - \mathbb{E}\{Z\}^2$$

Standard deviation

$$\Delta Z \equiv \sqrt{\text{Var}\{Z\}}$$

For our example, just need

$$\mathbb{E}\{Z^2\} = (+1)^2|\alpha|^2 + (-1)^2|\beta|^2$$
$$= |\alpha|^2 + |\beta|^2$$
$$\Rightarrow \text{Var}\{Z\} = |\alpha|^2 + |\beta|^2 - \left[|\alpha|^2 - |\beta|^2\right]^2$$
$$= 2|\alpha|^2|\beta|^2$$

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read section 3.3.2 on uncertainty principle

composite quantum systems

Cartesian product specifies all values of two classical bits c_0 & c_1

$Z_2 \equiv \{0,1\}$

$Z_2 \times Z_2 \equiv \{(0,0), (0,1), (1,0), (1,1)\}$

abbreviate as

$\{00, 01, 10, 11\}$

map to quantum states

by $00 \rightarrow |0\rangle|0\rangle$ or $|00\rangle$ (equivalently)

generally $c_1 c_0 \rightarrow |c_1 c_0\rangle$

By superposition, any linear combination of these states is possible

$\alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle$

also, $|\alpha|^2 + |\beta|^2 + |\gamma|^2 + |\delta|^2 = 1$

Cartesian product not sufficient for quantum states!

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need tensor product for
linear combinations (from linear algebra)

$$\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} \otimes \begin{bmatrix} a_2 \\ b_2 \end{bmatrix} = \begin{bmatrix} a_1 \begin{bmatrix} a_2 \\ b_2 \end{bmatrix} \\ b_1 \begin{bmatrix} a_2 \\ b_2 \end{bmatrix} \end{bmatrix}$$

convention
(TP ROL)

can show that

$$|00\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad |01\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad |10\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad |11\rangle = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

bits in ket index the 1 in vector

$$\Rightarrow \alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle = \begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{bmatrix}$$

can write this state in many different ways

$$\alpha|0\rangle\otimes|0\rangle + \beta|0\rangle\otimes|1\rangle + \gamma|1\rangle\otimes|0\rangle + \delta|1\rangle\otimes|1\rangle$$

OR

$$\alpha|0\rangle^A\otimes|0\rangle^B + \beta|0\rangle^A\otimes|1\rangle^B + \gamma|1\rangle^A\otimes|0\rangle^B + \delta|1\rangle^A\otimes|1\rangle^B$$

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Tensor product of ~~two~~ operators

$$A \otimes B = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \otimes \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} B & a_{12} B \\ a_{21} B & a_{22} B \end{bmatrix}$$

(TP ROL)

Best to just use

$$(A \otimes B)(|\psi\rangle \otimes |\phi\rangle)$$

$$= A|\psi\rangle \otimes B|\phi\rangle$$

example: flip 1^{st} bit

$$(X \otimes I)(\alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle)$$

$$= \alpha|10\rangle + \beta|11\rangle + \gamma|00\rangle + \delta|01\rangle$$

or flip second

$$(I \otimes X) \text{ '' } = \alpha|01\rangle + \beta|00\rangle + \gamma|11\rangle + \delta|10\rangle$$

can find matrix representation for $X \otimes I$ as

$$\begin{array}{c|cccc} & |00\rangle & |01\rangle & |10\rangle & |11\rangle \\ \hline \langle 00| & 0 & 0 & 1 & 0 \\ \langle 01| & 0 & 0 & 0 & 1 \\ \langle 10| & 1 & 0 & 0 & 0 \\ \langle 11| & 0 & 1 & 0 & 0 \end{array}$$

CNOT gate

$$|00\rangle \rightarrow |00\rangle$$
$$|01\rangle \rightarrow |01\rangle$$
$$|10\rangle \rightarrow |11\rangle$$
$$|11\rangle \rightarrow |10\rangle$$

can write as

$$|0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes X$$

No Cloning Theorem

- no universal copying machine in QM
- follows from linearity of QM

should do

$$U|\psi\rangle|0\rangle \rightarrow |\psi\rangle|\psi\rangle$$
$$= (\alpha|0\rangle + \beta|1\rangle)(\alpha|0\rangle + \beta|1\rangle)$$
$$= \alpha^2|00\rangle + \alpha\beta|01\rangle + \beta\alpha|10\rangle + \beta^2|11\rangle$$

can copy

$$U|0\rangle|0\rangle \rightarrow |0\rangle|0\rangle$$
$$U|1\rangle|0\rangle \rightarrow |1\rangle|1\rangle$$

But linearity implies

$$U(\alpha|0\rangle + \beta|1\rangle)|0\rangle \Rightarrow \alpha|00\rangle + \beta|11\rangle$$

Schmidt decomposition

Standard form for pure bipartite states

$|\psi\rangle^{AB}$ can always write as

$$|\psi\rangle^{AB} = \sum_i \lambda_i |i\rangle^A |i\rangle^B$$

where $\sum_i |\lambda_i|^2 = 1$

Proof. arbitrary state has form

$$|\psi\rangle^{AB} = \sum_{j,k} \alpha_{jk} \; |j\rangle^A |k\rangle^B$$

write coefficients in matrix

$$A = [\alpha_{jk}]$$

diagonalize A as $A = U \Lambda V$

$$\Rightarrow \alpha_{jk} = \sum_i u_{j,i} \lambda_i v_{i,k}$$

substitute

$$\sum_{j,k} \left(\sum_i u_{j,i} \lambda_i v_{i,k} \right) |j\rangle |k\rangle$$

$$= \sum_i \lambda_i \left(\sum_j u_{j,i} |j\rangle \right) \left(\sum_k v_{ik} |k\rangle \right)$$

$$= \sum_i \lambda_i |i\rangle |i\rangle$$