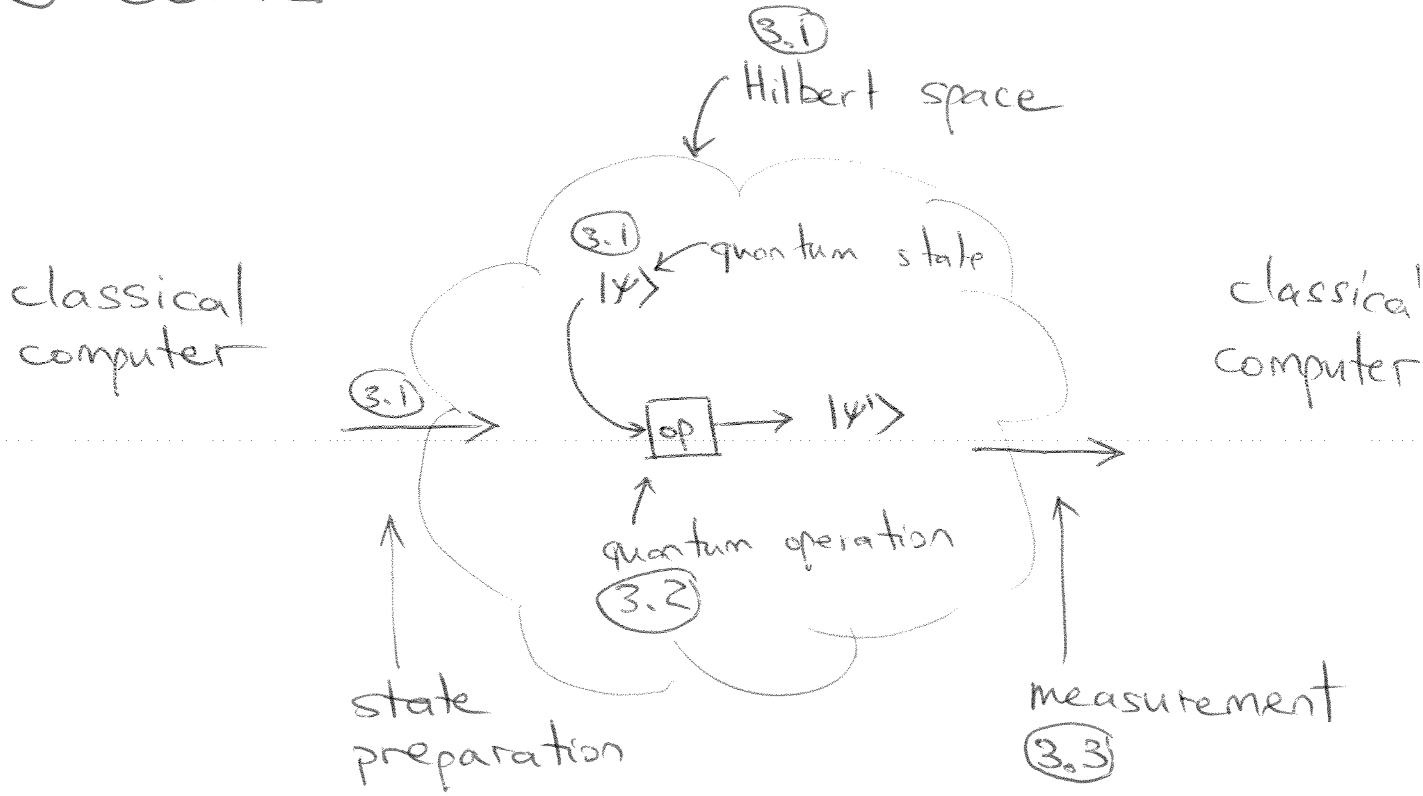
3. QUANTUM THEORY

① Context



(1) QUANTUM MECHANICS

→ need something <u>complex</u> & <u>vector-like</u> to explain physical experiments

→ originally used <u>wavefunctions</u> as the vector-like quantity

$$\left\{ f(x) \in \mathbb{C},\ x \in \mathbb{R} \;\middle|\; \int_{-\infty}^{\infty} |f(x)|^2\,dx = 1,\ f: \mathbb{R} \to \mathbb{C} \right\}$$

→ Matrix formulation for QM is just as good

↳ Linear Algebra

(1) Linear algebra: executive summary

⇒ linear

$$f(\alpha a + \beta b) = \alpha f(a) + \beta f(b)$$

→ α, β part of some <u>field</u>, ex: $\mathbb{R}$, $\mathbb{C}$

→ a, b part of a <u>vector space</u> V

- vectors
- matrices
- functions
- ...

→ f is some form of <u>transformation</u>

- matrix
- superoperator
- transf.

$$f \in L(V,V)$$

→ <u>Basis</u> for V is some set of $\dim(V)$ <u>orthonormal</u> vectors $\{\vec{e}_i\}_{i=1 \dots \dim(V)}$

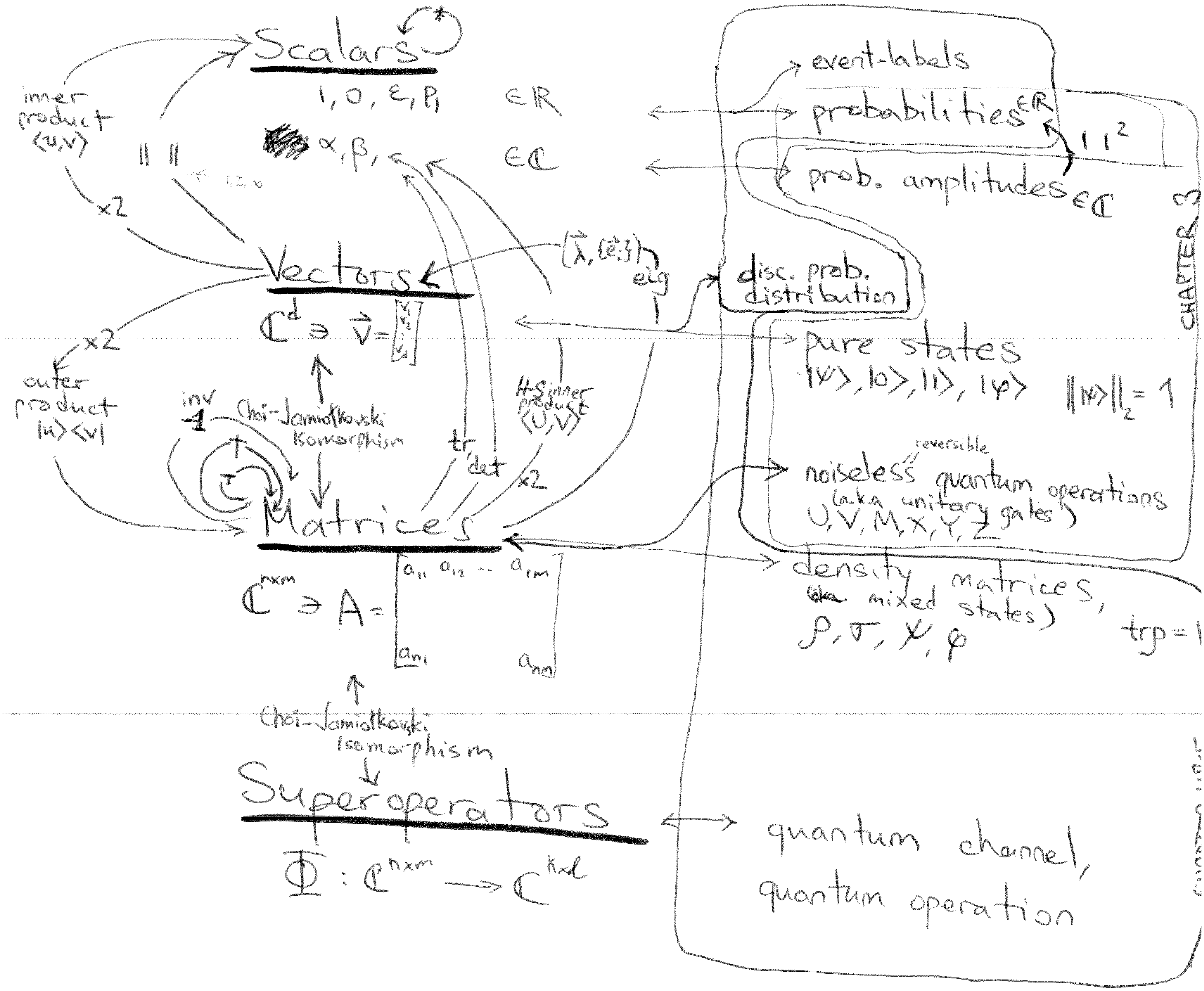
↳ $\langle e_i, e_j \rangle = \delta_{ij}$

→ <u>Inner product</u> i.e. dot-product, scalar-product

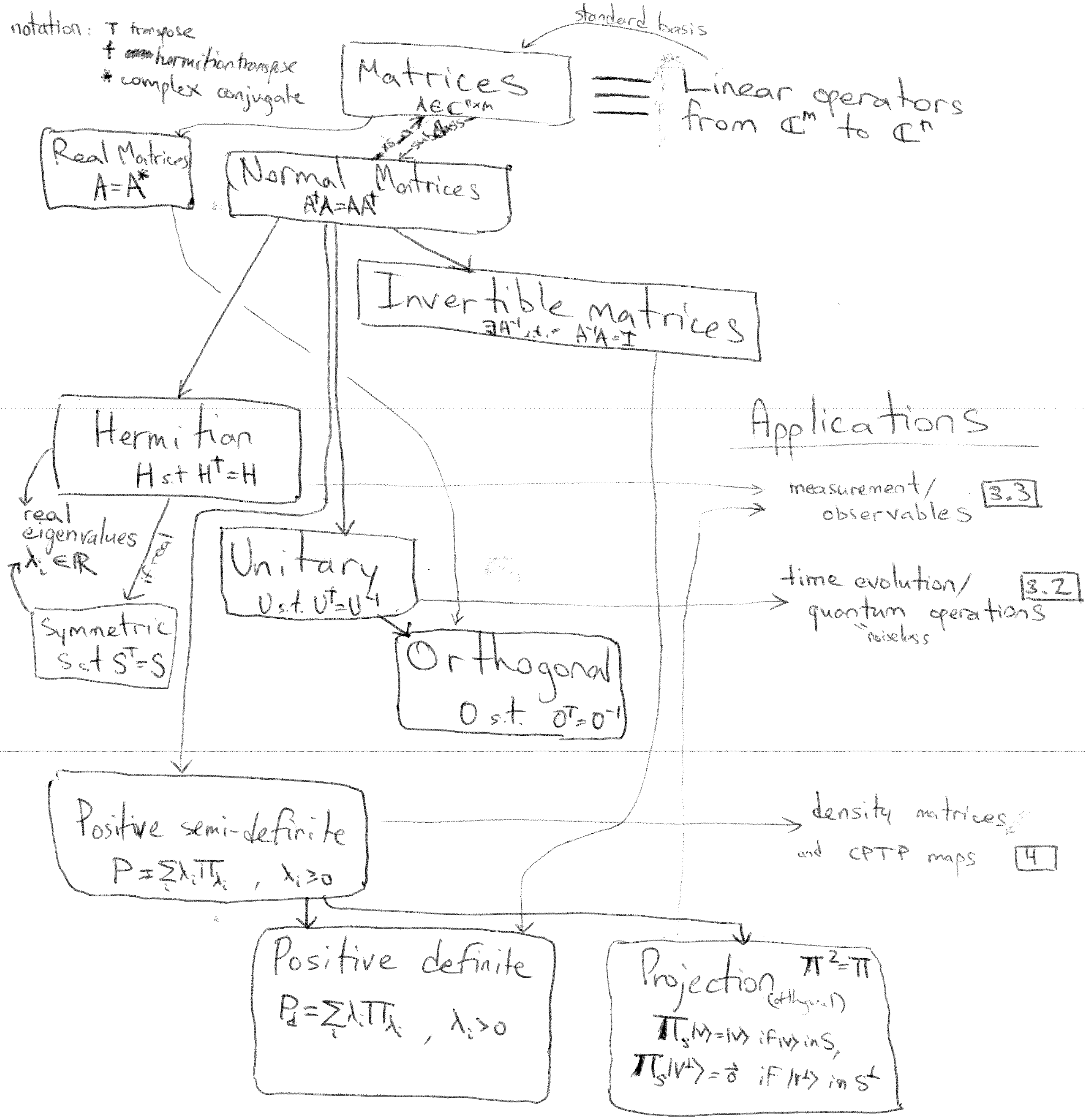
→ Killer app: to specify what f does need to specify $\{f(e_i)\}_{i=1 \dots \dim V}$

LINEAR ALGEBRA

APPLICATIONS TO MATRIX QUANTUM INFO...



Who is who in LA



(1) Definitions

(1) Quantum states live in <u>Hilbert space</u>

a complex Eucledian vector space with an inner product.

$\mathcal{H}, \mathcal{H}^d, \mathcal{X}, \mathcal{Y}, \mathcal{A}$

(2) Pure <u>quantum states</u> are unit-length vectors in Hilbert space. $\mathcal{H}^A$

$|\psi\rangle, |\varphi\rangle$

→ only dir matters, sometimes called

(3) For a d-dimensional Hilbert space $\mathcal{H}^d$, define the <u>standard basis</u>.

$$\{|0\rangle, |1\rangle, \ldots, |d-1\rangle\}$$

$$\begin{bmatrix}1\\0\\0\\\vdots\\0\end{bmatrix} \quad \begin{bmatrix}0\\1\\0\\\vdots\\0\end{bmatrix} \quad \begin{bmatrix}0\\0\\\vdots\\0\\1\end{bmatrix}$$

(4) For any vector $|\psi\rangle \in \mathcal{H}$, we define the <u>Hermitian transpose</u> as the combination of $|\psi\rangle$ to be its conjugate & transpose

$$|\psi\rangle = \begin{bmatrix}\alpha\\\beta\\\gamma\end{bmatrix} \xrightarrow{\dagger} \langle\psi| = [\alpha^* \ \beta^* \ \gamma^*]$$

DIRAC NOTATION → tired of writing $\vec{e}$

1 Vectors

Let $\alpha, \beta \in \mathbb{C}$ and $\vec{v}, \vec{w}, \vec{a}, \vec{b}$ be vectors in $\mathbb{C}^d$, then the Dirac notation for them is as follows:

$$\vec{v} \equiv |v\rangle \qquad \text{(called a "ket")}$$

The standard basis $\mathcal{B}_Z = \{\vec{e}_0, \vec{e}_1, \ldots, \vec{e}_{d-1}\}$:

$$\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \equiv |0\rangle, |1\rangle, \ldots, |d-1\rangle$$

$$[1, 0, \ldots, 0] \equiv \langle 0|$$

For a *qubit*, $d = 2$ we have:

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \equiv |0\rangle, |1\rangle$$

$$[1, 0],\ [0, 1] \equiv \langle 0|, \langle 1|$$

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} \equiv \alpha|0\rangle + \beta|1\rangle$$

Dagger ($\dagger$) is transpose (T) + complex conjugate ($*$):

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix}^\dagger = [\alpha^*, \beta^*] \equiv \alpha^*\langle 0| + \beta^*\langle 1|$$

$$(\vec{v}^T)^* = (\vec{v}^*)^T = \vec{v}^\dagger \equiv \langle v| = |v\rangle^\dagger \qquad \text{(called a "bra")}$$

We use the following inner product $(.,.)$:

$$\sum_{i=0}^{d-1} a_i^* b_i \equiv (\vec{a}, \vec{b}) \equiv \vec{a}^\dagger \vec{b} \equiv \langle a|b\rangle$$

$$(\vec{v}, \alpha\vec{a} + \beta\vec{b}) \equiv \alpha\vec{v}^\dagger\vec{a} + \beta\vec{v}^\dagger\vec{b} \equiv \alpha\langle v|a\rangle + \beta\langle v|b\rangle = \langle v|\big(\alpha|a\rangle + \beta|b\rangle\big)$$

$$(\alpha\vec{a} + \beta\vec{b}, \vec{w}) \equiv \alpha^*\vec{a}^\dagger\vec{w} + \beta^*\vec{b}^\dagger\vec{w} \equiv \alpha^*\langle a|w\rangle + \beta^*\langle b|w\rangle = \big(\alpha^*\langle a| + \beta^*\langle b|\big)|w\rangle$$

$$|\vec{v}| = \sqrt{\vec{v}^\dagger\vec{v}} \equiv \sqrt{\langle v|v\rangle} = ||v\rangle|$$

2 Different bases

Let $\vec{v} = [v_0, v_1, \ldots, v_{d-1}]^T$, where $v_i \in \mathbb{C}$ are the coefficients of $\vec{v}$ with respect to the standard basis $\mathcal{B}_Z = \{\vec{e}_i\}_{i=0,\ldots,d-1}$. We can calculate each coefficient using the inner product:

$$\underbrace{[0, \ldots,}_{i} 1, 0, \ldots] \begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_{d-1} \end{bmatrix} = \vec{e}_i^\dagger \vec{v} \equiv v_i = \langle i|v\rangle$$

$$\begin{aligned} \vec{v} = [v_0, v_1, \ldots, v_{d-1}]^T &\equiv v_0|0\rangle + v_1|1\rangle + \cdots + v_{d-1}|d-1\rangle \\ &= \langle 0|v\rangle|0\rangle + \langle 1|v\rangle|1\rangle + \cdots + \langle d-1|v\rangle|d-1\rangle \end{aligned}$$

The last expression explicitly shows that the basis $\mathcal{B}_Z = \{|i\rangle\}_{i=0,\ldots,d-1}$ was used. This comes in handy when using a different choice of basis like the Hadamard basis for example $\mathcal{B}_X = \{\vec{h}_0, \vec{h}_1\}$ when $d = 2$:

$$\frac{1}{\sqrt{2}}\vec{e}_0 + \frac{1}{\sqrt{2}}\vec{e}_1 = \vec{h}_0 \equiv |+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

$$\frac{1}{\sqrt{2}}\vec{e}_0 - \frac{1}{\sqrt{2}}\vec{e}_1 = \vec{h}_1 \equiv |-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$$

To get the coefficients of $\vec{v}$ with respect to the Hadamard basis:

$$\begin{aligned} \vec{v} = [v_+, v_-]_{\mathcal{B}_X}^T &\equiv v_+|+\rangle + v_-|-\rangle \\ &= \langle +|v\rangle|+\rangle + \langle -|v\rangle|-\rangle, \end{aligned}$$

where we had to specify the basis on the left. In the bra-ket notation, the positive coefficient w.r.t. $\mathcal{B}_X$ is $v_+ \equiv \langle +|v\rangle$ and the negative coefficient w.r.t. $\mathcal{B}_X$ is $v_- \equiv \langle -|v\rangle$.

Change of basis is simple in ket notation:

$$\begin{aligned} \vec{v} = [v_+, v_-]_{\mathcal{B}_X}^T &\equiv \langle +|v\rangle|+\rangle + \langle -|v\rangle|-\rangle \\ &= \langle +|\big(v_0|0\rangle + v_1|1\rangle\big)|+\rangle + \langle -|\big(v_0|0\rangle + v_1|1\rangle\big)|-\rangle \\ &= \big(v_0\langle +||0\rangle + v_1\langle +||1\rangle\big)|+\rangle + \big(v_0\langle -||0\rangle + v_1\langle -||1\rangle\big)|-\rangle \\ &= \underbrace{\frac{1}{\sqrt{2}}(v_0 + v_1)}_{v_+}|+\rangle + \underbrace{\frac{1}{\sqrt{2}}(v_0 - v_1)}_{v_-}|-\rangle, \end{aligned}$$

where v_0, v_1 were the coefficients of $\vec{v}$ in the standard basis.

(5) We use the standard <u>inner product</u> for $\mathcal{H}$

$$\langle \vec{u}, \vec{v} \rangle = \vec{u}^\dagger \vec{v}$$

$$= \begin{bmatrix} u_0^* & u_1^* & \cdots & u_{d-1}^* \end{bmatrix} \begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_{d-1} \end{bmatrix}$$

$$= \sum_{i=0}^{d-1} u_i^* v_i = \langle u | v \rangle$$

(6) In analogy with the classical bit, we define a quantum bit or <u>qubit</u> as a unit vector in $\mathcal{H}^2 \cong \mathbb{C}^2$.

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

$\alpha \in \mathbb{R}$, WLOG $\leftrightarrow$ global phase not important
$\beta \in \mathbb{C}$,

$$|\alpha|^2 + |\beta|^2 = 1$$

→ Observe that though there are 4 d.f. for vectors in $\mathbb{C}^2$, a qubit is actually 2 d.f.

$$4 \text{ d.f} - \underline{\alpha \text{ real}} - \|.\| = 1 = 2 \text{ d.f.}$$

→ For <u>qudit</u> $\underset{\text{d.f.}}{2 \cdot d} - \alpha \text{ real} - \|.\| = 1 = 2d - 2$ d.f.

Observe that

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

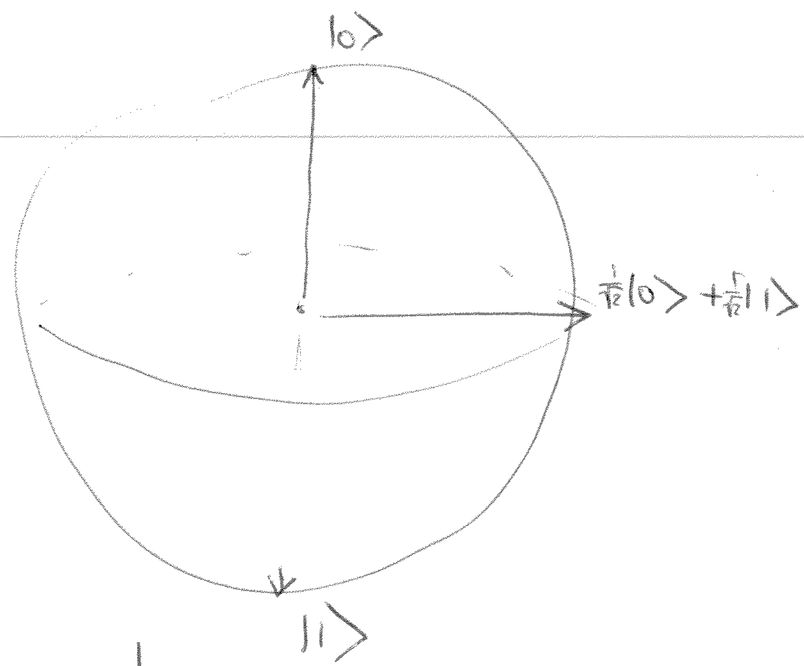
$$= \alpha|0\rangle + |\beta|e^{i\phi}|1\rangle$$

$$= \cos(\theta/2)|0\rangle + \sin(\theta/2)e^{i\phi}|1\rangle$$

We can identify the two angles

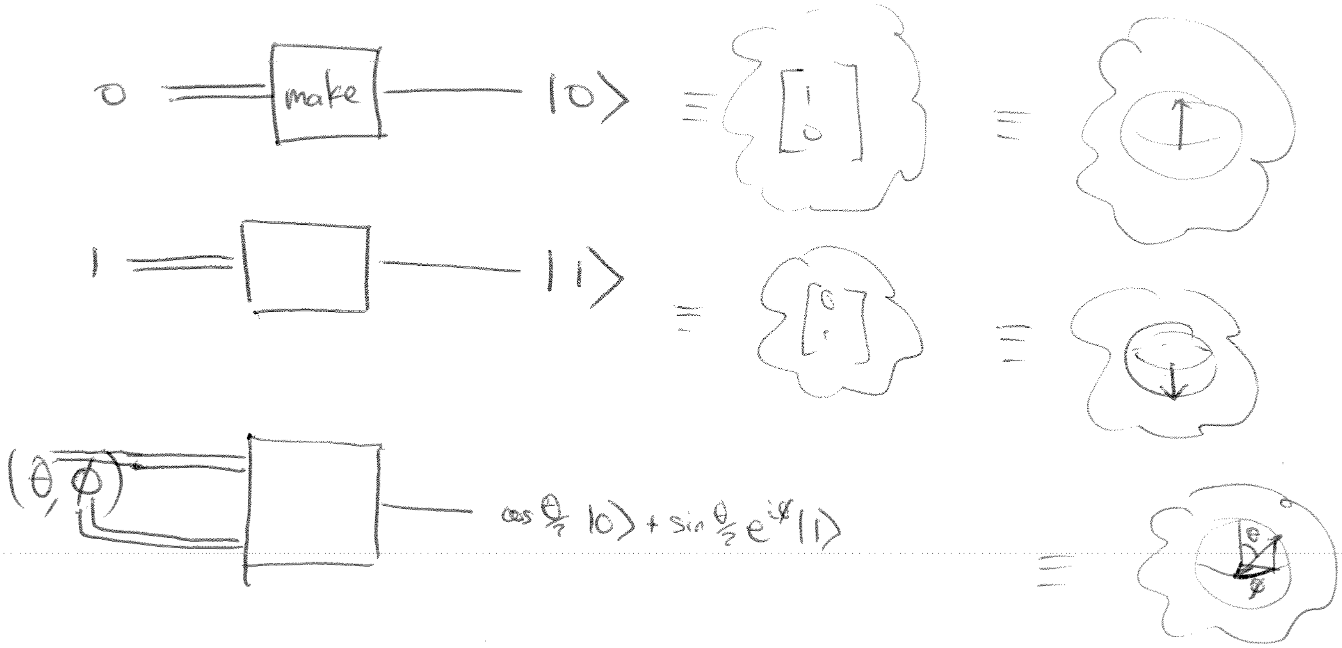
$0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$

as "polar coordinates" on the <u>Bloch sphere</u>



<u>note</u>: $|0\rangle$, $|1\rangle$ <u>are</u> orthogonal

① ~~[crossed out]~~ State preparation



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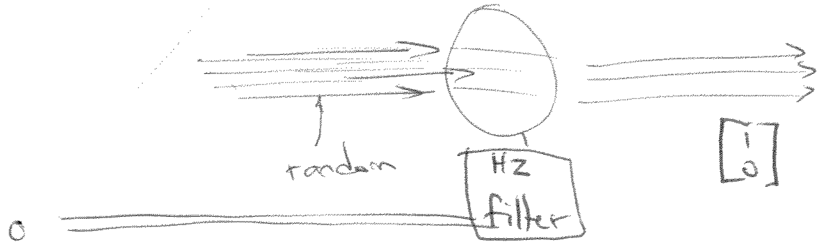
ex: light polarization

$\vec{Hz} = \begin{pmatrix}1\\0\end{pmatrix}$, $\vec{V} = \begin{pmatrix}0\\1\end{pmatrix}$

$\vec{+45^\circ} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$, $\vec{-45^\circ} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$

$\vec{LC} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}$, $\vec{RC} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}$

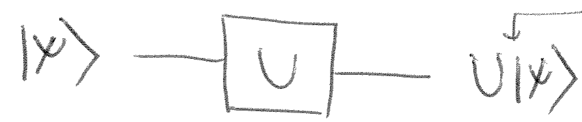


3.2 Quantum gates

→ noisy vs noiseless
↓
next class

(7) <u>Quantum operations</u> on pure quantum states are represented as <u>unitary</u> operators.

$U^\dagger U = \mathbb{1} = U U^\dagger$



matrix multiplication

→ this ensures states remain unit length

$\| \, |\psi\rangle \, \| = 1 \longrightarrow \| U|\psi\rangle \| = 1$

$\langle U|\psi\rangle, U|\psi\rangle \rangle =$

$= \langle \psi | U^\dagger U | \psi \rangle$

$= \langle \psi | \mathbb{1} | \psi \rangle = 1$

$\mathbb{1}$ idenity op.

(8) Define the phase flip operator, Z,

$$Z|0\rangle = |0\rangle$$

$$Z|1\rangle = -|1\rangle$$

and the not gate X

$$X|0\rangle = |1\rangle$$

$$X|1\rangle = |0\rangle$$

~~and the Hadamard gate $H|0\rangle = |+\rangle$, $H|1\rangle = |-\rangle$~~

→ If quantum states are vectors then quantum operations are matrices.

$$Z \begin{cases} \begin{bmatrix} & \\ & \end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix} \\ \begin{bmatrix} & \\ & \end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix} \end{cases}$$

in gener

$$\begin{bmatrix} & \\ & \end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} =$$

$$\begin{bmatrix} & \\ & \end{bmatrix}$$

→ in a different basis the same operator will correspond to different matrix

$|+\rangle \doteq \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

ex:

$$X|+\rangle = X\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right)$$

$$= |+\rangle$$

$$X|-\rangle = ($$

$$= |-\rangle$$

→ $|+\rangle$, $|-\rangle$ are eigenvectors of X with eigenvalues $+1$, -1.

→ eigenvectors are orth.

→ Define the <u>Hadamard basis</u> $\{|+\rangle, |-\rangle\}$.

to convert

$$\text{std basis} \xleftrightarrow{H} \text{hada basis}$$