Superadditivity of Classical Information

M. Hastings, “A Counterexample to Additivity of Minimum Output Entropy”

P. Shor, “Equivalence of Additivity Questions in Quantum Information Theory”

Have that in classical case, single letter formula exists for (block-coding) capacity, by additivity.

HSW Theorem: Maximum Holevo information gives classical capacity for quantum channel when *product states* are used.

$$C(\Lambda) = \lim_{n\rightarrow\infty} \max_{\{p_i^{(n)},\rho_{(n)}^i\}} \frac{1}{n}\Big[S\Big(\sum_i p_i^{(n)}\Lambda^{\otimes n}(\rho_{(n)}^i)\Big) - \sum_i p_i^{(n)} S(\Lambda^{\otimes n}(\rho_{(n)}^i))\Big]$$

If Holevo is additive, then it is a plausible formula for classical channel capacity of a quantum channel.

Equivalence of Additivity Conjectures (Shor, 2003)

- Additivity of minimum entropy output
- Additivity of Holevo capacity
- Additivity of Entanglement of Formation
- Strong superadditivity of Entanglement of Formation

$$E_f(\rho) = \min_{\mathcal{E}} p_i E_E(|\psi_i\rangle)$$

Hastings, 2008

- Minimum output entropy easier to work with than Holevo information:

$$H^{\min}(\mathcal{E}) = \min_{|\psi\rangle} H(\mathcal{E}(|\,\psi\rangle\langle\psi\,|))$$

- Constructs two channels such that minimum output entropy of product channel is less than sum of individual minimum output entropies (probabilities and unitaries are randomly chosen):

$$\mathcal{E}(\rho) := \sum_{i=1}^{D} P_i U_i^\dagger \rho U_i$$

$$\overline{\mathcal{E}}(\rho) := \sum_{i=1}^{D} P_i \overline{U}_i^\dagger \rho \overline{U}_i$$

The unitary operators are chosen from the unitary group of order N via the Haar measure.
The probabilities are chosen via:

$$P_i = l_i^2 / L^2$$

for distributions

$$P(l_i) \propto l_i^{2N-1} \exp(-N D l_i^2)$$

This is the same as choosing the l's from a Gaussian random variable on N complex dimensions.

Main Result

We have, for unitaries and probabilities selected from some distribution,

$$H^{\min}(\mathcal{E} \otimes \overline{\mathcal{E}}) < H^{\min}(\mathcal{E}) + H^{\min}(\overline{\mathcal{E}}) = 2H^{\min}(\mathcal{E})$$

Which implies via equivalence of additivity conjectures that Holevo information is non-additive.

Recall the maximally mixed state:

$$| \psi_{ME}\rangle := (1/\sqrt{N_D}) \sum_{\alpha=1}^{N_D} | \alpha\rangle \otimes | \alpha\rangle$$

Lemma 1: Have that

$$H(\mathcal{E} \otimes \overline{\mathcal{E}}(| \psi_{ME}\rangle\langle\psi_{ME} |)) \leq 2 \ln D - \ln D/D$$

Hence, minimum output entropy is also bounded.

Lemma 2: Hard Bound

If $\{U_i\}$ *are chosen randomly from the set of unitary matrices of size* N*, and* $\{l_i\}$ *are chosen randomly as above, then*

$$\mathbb{P}(H^{\min}(\mathcal{E}^c) < \ln(D) - \delta S^{\max}) < 1$$

for an appropriate choice of c_1 *and* p_1 *(in the definition of* $\delta S^{\max}$*) and for* $1 \ll D \ll N$*. This in particular implies that we obtain a counter-example to the additivity conjecture once* D *and* N *are sufficiently large, i.e. there exist* $\hat{D}$ *and* $\hat{N}$ *such that for* $\hat{D} < D$ *and* $\hat{N} < N$ *the above holds and a counter-example to additivity exists.*

$$\delta S^{\max} = c_1/D + p_1(D)\mathcal{O}(\sqrt{\ln(N)/N})$$

Conclusion

- Proof is probabilistic, and hence nonconstructive. Have lower bounds on size of D.
- The Holevo information is additive for certain classes of channels: entanglement-breaking and depolarizing channels are Holevo additive.
- Alternative proofs exist; i.e. Via Dvoretzky's theorem in functional analysis.

Open Problems

- A (computable) formula for the classical capacity that accounts for entanglement across input spaces

- Shor asked about the additivity of the following:

$$\max \lambda H(\rho) - H(\mathcal{E}(\rho))$$