M. Hastings, “A Counterexample to Additivity of Minimum Output Entropy”

P. Shor, “Equivalence of Additivity Questions in Quantum Information Theory”
Have that in classical case, single letter formula exists for (block-coding) capacity, by additivity.

HSW Theorem: Maximum Holevo information gives classical capacity for quantum channel when *product states* are used.

\[
C(\Lambda) = \lim_{n \to \infty} \max_{\{p_i^{(n)}, \rho_i^{(n)}\}} \frac{1}{n} \left[ S\left( \sum_i p_i^{(n)} \Lambda^{\otimes n}(\rho_i^{(n)}) \right) - \sum_i p_i^{(n)} S(\Lambda^{\otimes n}(\rho_i^{(n)})) \right]
\]

If Holevo is additive, then it is a plausible formula for classical channel capacity of a quantum channel.
Equivalence of Additivity Conjectures (Shor, 2003)

- Additivity of minimum entropy output
- Additivity of Holevo capacity
- Additivity of Entanglement of Formation
- Strong superadditivity of Entanglement of Formation

\[ E_f(\rho) = \min_{\varepsilon} p_i E_E(\ket{\psi_i}) \]
Hastings, 2008

- Minimum output entropy easier to work with than Holevo information:

\[ H_{\text{min}}(\mathcal{E}) = \min_{|\psi\rangle} H(\mathcal{E}(|\psi\rangle\langle\psi|)) \]

- Constructs two channels such that minimum output entropy of product channel is less than sum of individual minimum output entropies (probabilities and unitaries are randomly chosen):

\[ \mathcal{E}(\rho) := \sum_{i=1}^{D} P_i U_i^\dagger \rho U_i \]

\[ \overline{\mathcal{E}}(\rho) := \sum_{i=1}^{D} P_i \overline{U}_i^\dagger \rho \overline{U}_i \]
The unitary operators are chosen from the unitary group of order $N$ via the Haar measure. The probabilities are chosen via:

$$P_i = \frac{l_i^2}{L^2}$$

for distributions

$$P(l_i) \propto l_i^{2N-1} \exp(-NDl_i^2)$$

This is the same as choosing the $l$'s from a Gaussian random variable on $N$ complex dimensions.
Main Result

We have, for unitaries and probabilities selected from some distribution,

$$H_{\text{min}}(E \otimes \bar{E}) < H_{\text{min}}(E) + H_{\text{min}}(\bar{E}) = 2H_{\text{min}}(E)$$

Which implies via equivalence of additivity conjectures that Holevo information is non-additive.
Recall the maximally mixed state:

\[ | \psi_{ME} \rangle := \frac{1}{\sqrt{N_D}} \sum_{\alpha=1}^{N_D} | \alpha \rangle \otimes | \alpha \rangle \]

Lemma 1: Have that

\[ H(\mathcal{E} \otimes \mathcal{E}(| \psi_{ME} \rangle \langle \psi_{ME} |)) \leq 2 \ln D - \ln D/D \]

Hence, minimum output entropy is also bounded.
Lemma 2: Hard Bound

If \( \{U_i\} \) are chosen randomly from the set of unitary matrices of size \( N \), and \( \{l_i\} \) are chosen randomly as above, then

\[
\mathbb{P}(H_{\min}(\mathcal{E}^c) < \ln(D) - \delta S_{\max}) < 1
\]

for an appropriate choice of \( c_1 \) and \( p_1 \) (in the definition of \( \delta S_{\max} \)) and for \( 1 \ll D \ll N \). This in particular implies that we obtain a counter-example to the additivity conjecture once \( D \) and \( N \) are sufficiently large, i.e. there exist \( \hat{D} \) and \( \hat{N} \) such that for \( \hat{D} < D \) and \( \hat{N} < N \) the above holds and a counter-example to additivity exists.

\[
\delta S_{\max} = c_1/D + p_1(D)O(\sqrt{\ln(N)/N})
\]
Conclusion

- Proof is probabilistic, and hence nonconstructive. Have lower bounds on size of $D$.
- The Holevo information is additive for certain classes of channels: entanglement-breaking and depolarizing channels are Holevo additive.
- Alternative proofs exist; i.e. Via Dvoretzky's theorem in functional analysis.
Open Problems

• A (computable) formula for the classical capacity that accounts for entanglement across input spaces

• Shor asked about the additivity of the following:

\[
\max \lambda H(\rho) - H(\mathcal{E}(\rho))
\]