Capacity Formula for Quantum Channels

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Three Qbit Bit-Flip Codes

- Alice wants to send Bob a state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ through a bit-flip channel.

- Let channel flip the bit with probability $0 < p < 0.5$

- Alice compute $|\psi\rangle|00\rangle = \alpha|000\rangle + \beta|100\rangle$

- Alice apply CNot gate from 1st qubit to 2nd and 3rd qubits producing the state $\alpha|000\rangle + \beta|111\rangle$

- Alice send Bob the state through the channel

Degenerate Codes

- Degeneracy is a property of Codes and a family of errors it design to correct. More formally a code degenerately correct a set of errors $\underline{E}$ if in addition to correcting $\underline{E}$, there are multiple errors in $\underline{E}$ that are mapped to same syndrome.

Degradable Channels

•A channel is degradable if there exist a map

$$T^{B \to E}$$

such that for any input state

$$N_c^{A' \to E}(\rho) = T^{B \to E}\left(N^{A' \to B}(\rho)\right)$$

Properties of Degradable Channels

- The Capacity formula for a degradable channel is

$$C_q = Q(N) = max_{\phi^{AA'}} I(A>B)_{\rho^{AB}}$$

- For any two degradable channels, coherent information is additive

$$Q(N_1 \otimes N_2) = Q(N_1) + Q(N_2)$$

Pauli Channels

- A Pauli channel maps:

$$\rho \rightarrow \left(1 - p_x - p_y - p_z\right)\rho + p_x X \rho X + p_y Y \rho Y + p_z Z \rho Z$$

- A two Pauli channel is a special case of above:

$$\rho \rightarrow \left(1 - p\right)\rho + \left(p X \rho X + p Z \rho Z\right)$$

Quantum Capacity of a Channel

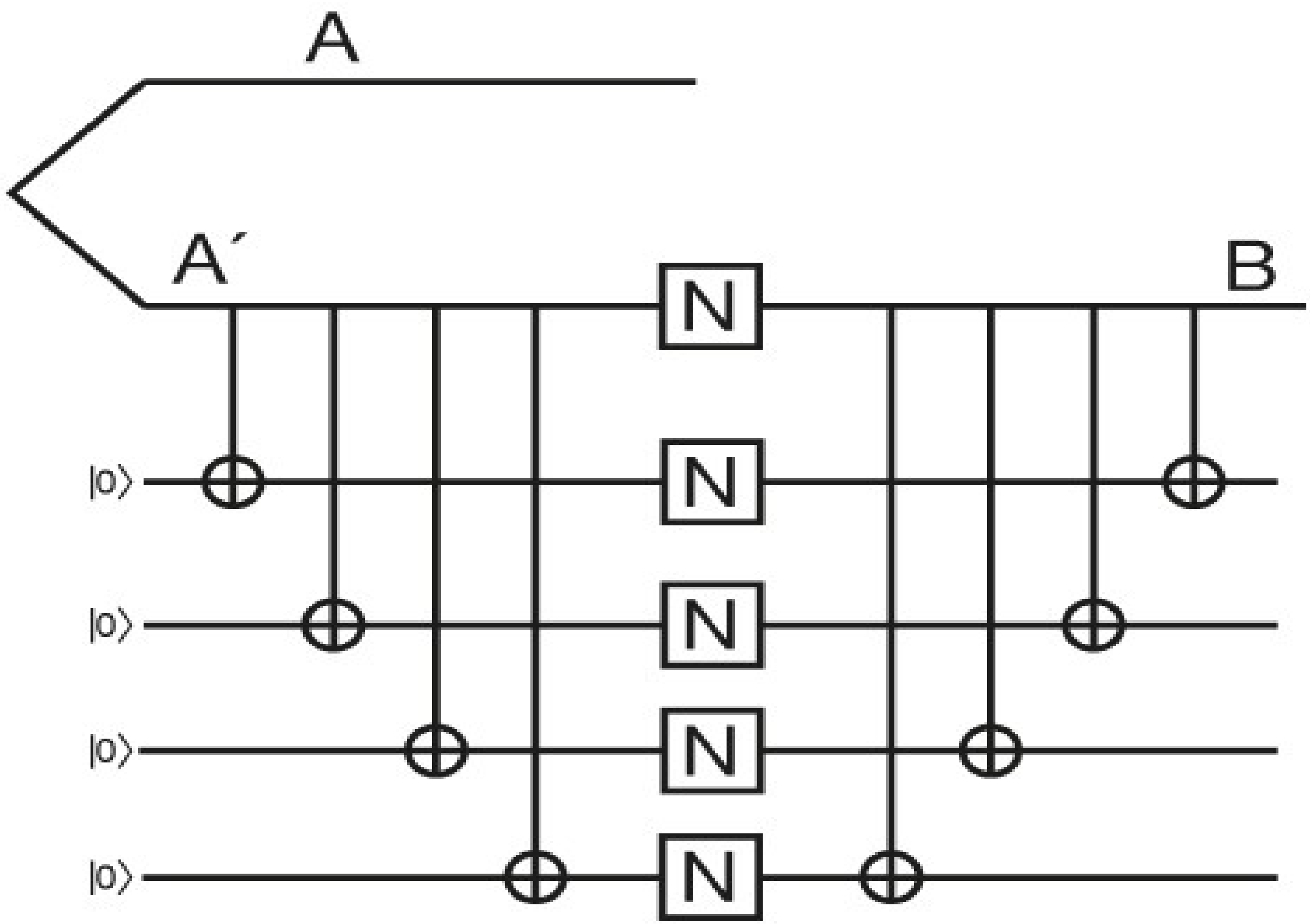
- In quantum world one would hope that the quantum capacity formula C_q is given by

$$C_q = Q(N) = max_{\phi^{AA'}} I(A > B)_{\rho^{AB}}$$

- This rate can be achieve using a random code on typical subspaces.

- However this is not optimal in genral.

- Using degenerate codes Shor and others, shows that $C_q > Q(N)$ (for depolarizing channel).
- For $p=0.189, Q(N)=0$ and then $\frac{1}{5} Q(N^{\otimes 5}) > 0$



Correct Characterization C_q

- The correct characterization of quantum capacity is given by

$$\lim_{n \to \infty} \frac{1}{n} Q(N^{\otimes n})$$

- However the rate they obtain was slightly greater than

$$Q(N) = max_{\phi^{AA'}} I(A > B)_{\rho^{AB}}$$

Classes of Capacity for Quantum Channels

- The correct characterization of quantum capacity is not very useful. But there are cases where do we have a useful formula.

- We can divide the capacity of quantum channels into three classes.

1) Classical Quantum Capacity.
2) Entanglement Assisted Capacity.
3) Quantum Capacity.

Classes of Quantum Capacity

Classical Capacity	Input	Output	Formula
	C	q	Yes
	q	c	Yes
	q	q	No
Entanglement-Assisted Classical Capacity	q	q	Yes
Quantum Capacity	q	q	No (except degradable)

S.S.D Result

- Recall that shor, Smolin, DiVincenzo show that $C_q > Q(N)$. However difference between was extremely small.
- Smith and Smolin shows for most pauli channels that the difference between C_q and $Q(N)$ is substantial.
- They showed this by creating massive degenerate codes

Non-degradable Channels

- From above discussion one may think:

1) Only degradable channel's capacity can be characterized by single letter formula.

2) There may be no single letter formula in general for non-degradable channels.

3) Capacity of any non-degradable channel cannot be given by coherent information.

Non-Degrability of a Channel

- Smith and Smolin propose a general method for showing that a channel is not degradable

- Based on this method they proved that two Pauli channel is not degradable.

- Yet for two Pauli channel the degenerate codes did not perform any better than non-degenerate codes. In both cases they obtain the rate $Q(N) = max_{\phi^{AA'}} I(A > B)_{\rho^{AB}}$

Conclusion

- A single letter formula for the Capacity is still possible. But what is not possible in general that formula is given by channels coherent information.
- It may also be the case that for some non-degradable channels channels coherent information may fully characterize its capacity.

Bit-Flip Codes

- Bob receives 3 qubits. Their state is one of the following (table). Bob prepared ancilla qubits $|00\rangle$ and carried out Cnot from 1st and 2nd received qubits to 1st ancilla and 1st and 3rd to 2nd ancilla

	Probability	Syndrome	Operation
$\alpha\|000\rangle+\beta\|111\rangle$	$1-p^3$	$\|00\rangle$	$I\otimes I\otimes I$
$\alpha\|100\rangle+\beta\|011\rangle$	$p(1-p)^2$	$\|11\rangle$	$X\otimes I\otimes I$
$\alpha\|010\rangle+\beta\|101\rangle$	$p(1-p)^2$	$\|10\rangle$	$I\otimes X\otimes I$
$\alpha\|001\rangle+\beta\|110\rangle$	$p(1-p)^2$	$\|01\rangle$	$I\otimes I\otimes X$
$\alpha\|110\rangle+\beta\|001\rangle$	$p^2(1-p)$	$\|01\rangle$	?
$\alpha\|101\rangle+\beta\|010\rangle$	$p^2(1-p)$	$\|10\rangle$	?
$\alpha\|011\rangle+\beta\|100\rangle$	$p^2(1-p)$	$\|11\rangle$	?
$\alpha\|111\rangle+\beta\|000\rangle$	p^3	$\|00\rangle$	?

Complementary Channels

- Let

$$N^{A'\to B}$$

be a noisy channel with isometric extension

$$U_N^{A'\to BE}$$

- A complementary channel is given by

$$N_c^{A'\to E}(\rho) = Tr_B(U_N^{A'\to BE})$$

Capacity Of a Channel

- Capacity of a channel is the tightest upper bound on the amount of information that can reliably transmitted over a channel.

- Classical capacity formula is given by

$$C = max_x I(X, N(X))$$

- The most fundamental problem of information is to obtain a formula for the capacity of a Noisy Channel.

Method for Proving Non-Degeradability

- Let $N^{A'\to B}$ be a channel and $N_c^{A'\to E}$ be the complementary channel.
- Compute the map $T^{B\to E}$ such that $N\,T = N^C$
- Compute Choi matrix of T
- If Choi matrix is not completely positive then channel is not-degradable.