Entanglement-Enhanced Classical Communication over a Noisy Channel

R. Prevedel, Y. Lu, W. Matthews, R. Kaltenback, and K.J. Resch

(October 2010)

Introduction

- We know that entanglement cannot increase the capacity of any classical channel in the sense of Shannon (why?)
- However, **the one-shot capacity** of a classical channel can be increased when the sender and receiver share entanglement
- In this paper, it was shown that given a single use of a particular classical noisy channel, a single bit can be transmitted with higher success probability when the sender and receiver shared entanglement compared to the best possible strategy when they do not. Furthermore, a physical experiment was performed that implemented the protocol and obtained results very close to the predicted theoretical values.

I.I.D. vs. one-shot

Independent and identically distributed (I.I.D.) protocol:

- Unlimited number of channel uses
- Success rate = 100% in the asymptotic limit (probability of error vanishes as n becomes large)

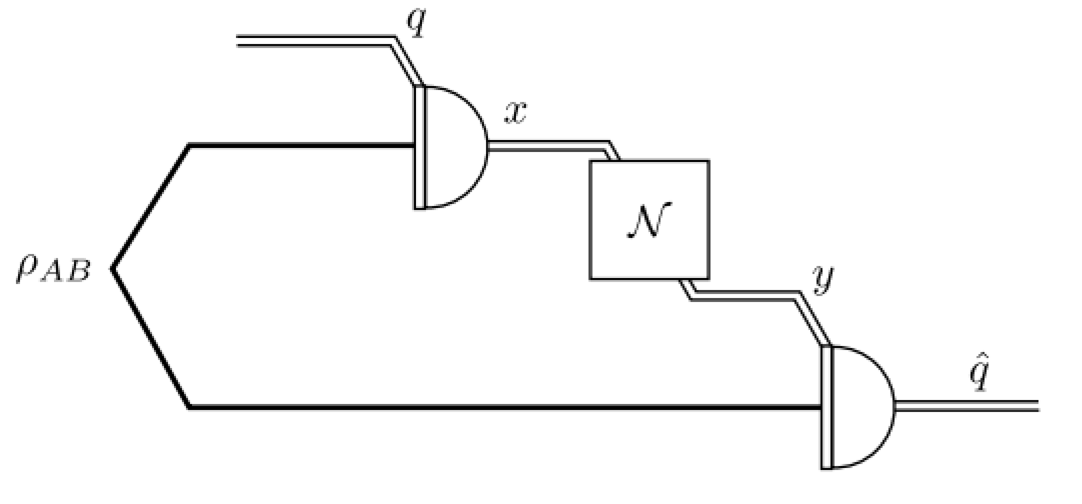
One-shot protocol:

- Only one use of the channel
- Success rate not specified but is generally < 100% unless it is a trivial noiseless channel

When can each protocol be used?

Basic Entanglement-Assisted Protocol

- In this protocol, message **q** is sent from Alice to Bob with a single use of a classical channel **N** while they share an entangled state **pAB**

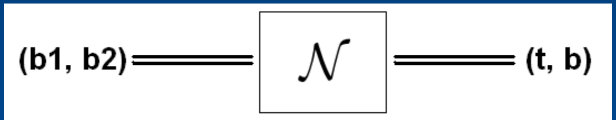


- We will now define the classical channel **N** and demonstrate both its maximal classical and quantum success rate

The Classical Channel

We will be using a particular classical channel that will:

- Take as input two classical bits: ***(b1, b2)***
- Output a trit and a bit ***(t, b)***



There are three equiprobable values for the output pair ***(t, b)***:

- *(1, **b1**)*, where the bit **b1** is the first input bit
- *(2, **b2**)*, where the bit **b2** is the second input bit
- *(P, **b1+b2** (mod 2))*, where the the bit in the second position is as the parity of both input bits.

So basically, our channel takes as input two classical bits, and transforms them into a trit and a bit, where the output trit indicates which of the 3 equiprobable possibilities was selected, while the output bit is the actual possibility value.

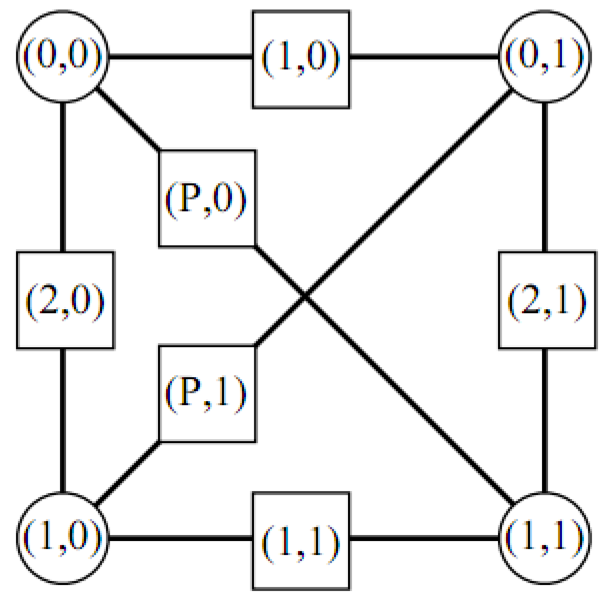
Classical channel continued

- In order to better illustrate the protocol, consider its table and graph:

(a)

Output \ Input	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
$(1,0)$	$1/3$	$1/3$	0	0
$(1,1)$	0	0	$1/3$	$1/3$
$(2,0)$	$1/3$	0	$1/3$	0
$(2,1)$	0	$1/3$	0	$1/3$
$(P,0)$	$1/3$	0	0	$1/3$
$(P,1)$	0	$1/3$	$1/3$	0

(b)

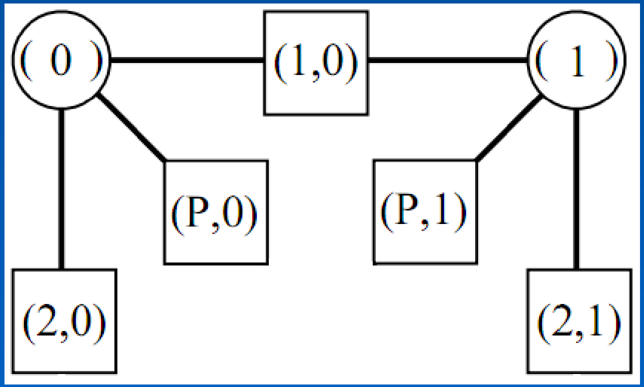


- Note that from any input, there are always three possible outputs

Maximum Success Probability (classical case)

The maximum success probability for sending one bit with a single use of the channel if only classical communication is used is 5/6:

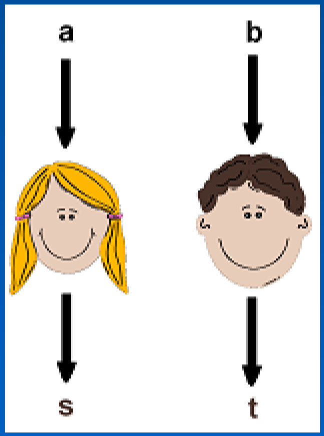
- We have 4 input states (2 * 2) and 6 output states (3 * 2)
- Given that we only need to transmit a single bit, we can use codewords to map our input two bit state into a single bit state:
- Now our graph maps an input bit state into 5 output states
- The result is that for every input state, 2 out of 3 output states are non-confusable, and the last output state has a probability of ½, which leads to a probability of: 2/3 + 1/3*½ = $5/6 \approx 0.83$



CHSH game (overview)

For the proof for the quantum case, we need to understand the **CHSH-inequality**. In order to do so, let’s consider a fictional game:

- Alice and Bob are given input bits **a** and **b**, respectively
- Alice and Bob then return output bits **s** and **t**, respectively, after performing some operation on their bit
- They win the game if:
  s + **t** (mod 2) = **ab**
- What strategy should they use?



CHSH game (classical strategy)

Table of results for the CHSH game:

(s,t) \ (a,b)	(0, 0)	(0, 1)	(1, 0)	(1, 1)
(0, 0)	WIN	WIN	WIN	LOSE
(0, 1)	LOSE	LOSE	LOSE	WIN
(1, 0)	LOSE	LOSE	LOSE	WIN
(1, 1)	WIN	WIN	WIN	LOSE

- Note that the first and last row have probability of winning of ¾, while it is only ¼ for the middle rows
- Thus, the best classical strategy for them to win is by setting **s** = **t**, thereby selecting the first and last rows
- The resulting probability of winning becomes ¾ = 75%
- Leads to the CHSH-inequality:

  | E(-1,-1) - E(-1,1) + E(1,-1) + E(1,1) | <= 2

CHSH game (quantum strategy)

- For the quantum strategy, assume that Alice and Bob share an entangled qubit: $|\Phi^-\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle - |11\rangle\right)$
- If we measure in the following basis:

$$X_0^a = |\phi_a(0)\rangle\langle\phi_a(0)|$$
$$X_1^a = |\phi_a(\pi/4)\rangle\langle\phi_a(\pi/4)|$$
$$Y_0^b = |\phi_b(\pi/8)\rangle\langle\phi_b(\pi/8)|$$
$$Y_1^b = |\phi_b(-\pi/8)\rangle\langle\phi_b(-\pi/8)|$$

where:

$$|\phi_0(\theta)\rangle = cos(\theta)|0\rangle + sin(\theta)|1\rangle$$
$$|\phi_1(\theta)\rangle = -sin(\theta)|0\rangle + cos(\theta)|1\rangle$$

- The measurement result will be: $\frac{1}{2}\mathrm{Tr}(X_s^a Y_t^b)$
- This will yield a winning rate of cos^2(PI / 8), which is about 85%
- This will also violate the CHSH-inequality stated previously

Maximum Success Probability (quantum case)

The maximum success probability for sending one bit with a single use of the channel in the quantum case will be: $(2+2^{-1/2})/3$

which is about 90.2 %:

- Taking advantage of entanglement, Alice and Bob will now share an entangled state: $|\Phi^+\rangle = (|HH\rangle + |VV\rangle)/\sqrt{2}$
- Alice and Bob will perform measurements in one of two basis, which are the measurements that lead to a maximal violation of the CHSH inequality:

Alice:
$$|\pi/4\rangle\langle\pi/4| - |3\pi/4\rangle\langle 3\pi/4|$$
$$|0\rangle\langle 0| - |\pi/2\rangle\langle\pi/2|$$

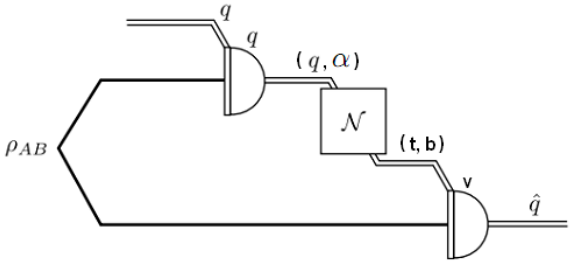
Bob:
$$|\pi/8\rangle\langle\pi/8| - |5\pi/8\rangle\langle 5\pi/8|$$
$$|3\pi/8\rangle\langle 3\pi/8| - |7\pi/8\rangle\langle 7\pi/8|$$

Thus, “q + v (mod 2) = ab” holds with probability w = cos^2(PI / 8)

Maximum Success Probability (quantum case)

The resulting table of results will be:

t	b	Bob chooses $v =$	β	$\hat{q}$	X_{on}	Z_{on}
1	q	irrelevant	n/a	b	n/a	n/a
2	α	1	$q \oplus \alpha$	$b \oplus \beta$	$1 \oplus b$	b
P	$q \oplus \alpha$	0	α	$b \oplus \beta$	b	b

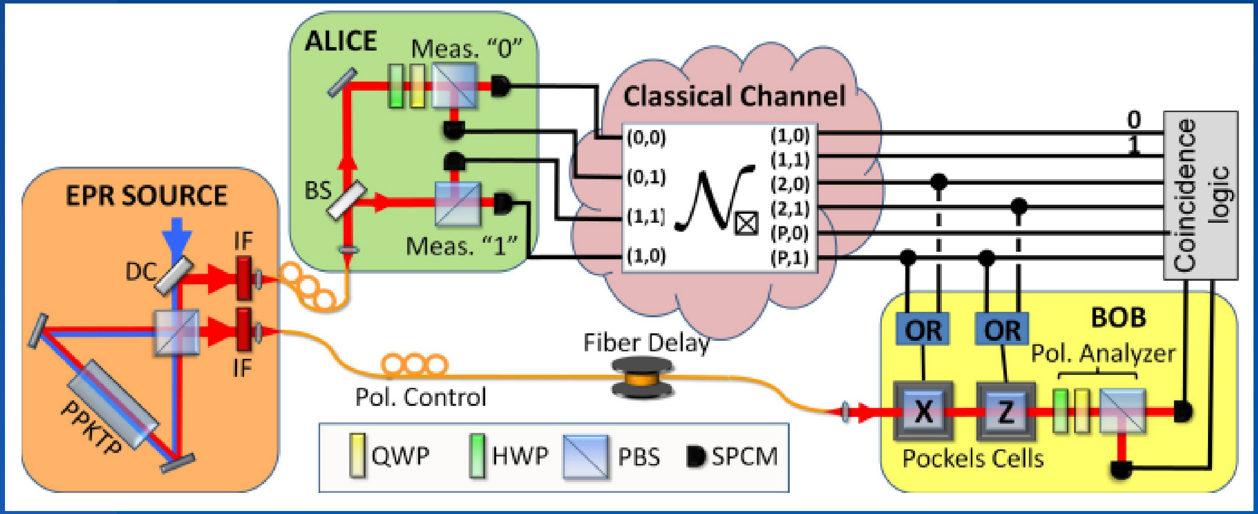


$\alpha \oplus \beta = qv$

- In the following table, we are assuming that we are always winning the CHSH-inequality game, which is only true for w = $(1+2^{-1/2})/2$
- Because the CHSH-inequality game is irrelevant for t = 1, one third of the time we are guaranteed to succeed. The final probability for winning will thus be: $\omega + (1-\omega)/3 = (2+2^{-1/2})/3$ which is about 90.2 %

Physical experiment

A physical experiment was also conducted by the authors of the paper using the following setup:



The resulting success rate obtained was $P_{\text{exp}} = 0.891 \pm 0.002$
which is very close to the theoretically rate $P_{\text{th}} = (2 + 2^{-1/2})/3 \approx 0.902$

Conclusion

In conclusion:

- The authors demonstrated the superiority of a quantum strategy over a classical one for the case of a one-shot protocol
- They also demonstrated that it is quite feasible experimentally, as an equivalent I.I.D. protocol would currently be almost impossible to implement given the current stage of experimental quantum physics, notably because of the huge amount of quantum channels that would need to be implemented
- It would be interesting to know what is the maximal amount of boost that is possible using entanglement over a purely classical strategy for an arbitrary noisy quantum channel using a one-shot protocol