A discussion of Information-Capacity description of spin-chain correlations

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Spin Chain Quantum Channel

- Originally proposed by Sougato Bose
- Spin chains are made up of some number of spins coupled together by some time-independent Hamiltonian.
- These spin-chains allow for the propagation of quantum states.

The Model

- N coupled spins, default state σ_0
- 2 Registers at either side of the spin chain, each of size k
- State ρ_A encoded on one register and some time later, decoded as ρ_B on other side.
- State of chain at time t: $R(t) = U(t)(\rho_A \otimes \sigma_0)U^\dagger(t)$
- Mapping: $\rho_A \to \mathcal{M}(\rho_A) \equiv \rho_B(t) = \mathrm{Tr}^{(B)}[U(t)(\rho_A \otimes \sigma_0)U^\dagger(t)]$

A Solvable Model

- Reading from register disturbs state so channel cannot be reused - difficult to solve
- Define a model for which meaningful capacities can be obtained

$$|j\rangle \equiv |\downarrow\downarrow\cdots\downarrow\uparrow\downarrow\cdots\downarrow\rangle$$

$$|\phi_1\rangle_A \equiv \sum_{j=1}^{k} c_j |j\rangle$$

$$|\Psi\rangle_A \equiv \alpha|\Downarrow\rangle_A + \beta|\phi_1\rangle_A$$

$$|\Psi(t)\rangle \equiv \alpha|\Downarrow\rangle + \beta \sum_{j'=1}^{N}\sum_{j=1}^{k} c_j f_{j',j}(t)|j'\rangle$$

Amplitude Damping Channel

- Looking at state on the receiving end, we see it is described by an amplitude damping channel!

$$\rho_B(t) = (|\alpha|^2 + (1-\eta)|\beta|^2)|\Downarrow\rangle_B\langle\Downarrow| + \eta|\beta|^2|\phi_1'\rangle_B\langle\phi_1'| + \sqrt{\eta}\alpha\beta^*|\Downarrow\rangle_B\langle\phi_1'| + \sqrt{\eta}\alpha^*\beta|\phi_1'\rangle_B\langle\Downarrow|$$

$$\rho' = \mathcal{D}_\eta(\rho)$$

$$\begin{aligned} A_0 &= |0\rangle\langle 0| + \sqrt{\eta}\,|1\rangle\langle 1|\,, \\ A_1 &= \sqrt{1-\eta}\,|0\rangle\langle 1|\,. \end{aligned}$$

Calculating Capacity

$$\mathcal{D}_{\eta'}\left(\mathcal{D}_\eta(\rho)\right) = \mathcal{D}_{\eta\eta'}(\rho)$$

$$\mathcal{D}_\eta(\rho) \equiv \mathrm{Tr}_C[V\,(\rho \otimes |0\rangle_C\langle 0|)\,V^\dagger]$$

$$\tilde{\mathcal{D}}_\eta(\rho) \equiv \mathrm{Tr}_A[V\,(\rho \otimes |0\rangle_C\langle 0|)\,V^\dagger]$$

$$\tilde{\mathcal{D}}_\eta(\rho) = S\,\mathcal{D}_{(1-\eta)/\eta}\left(\mathcal{D}_\eta(\rho)\right)\;S$$

$$V \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{\eta} & \sqrt{1-\eta} & 0 \\ 0 & -\sqrt{1-\eta} & \sqrt{\eta} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

A complementary channel is defined, and it is demonstrated that a map exists between ADC and it's complement, sufficient condition for Q being achieved by single channel use.

Quantum Capacity

$$Q_n \equiv \max_{\rho \in \mathcal{H}^{\otimes n}} \left\{ S(\mathcal{M}^{\otimes n}(\rho)) - S((\mathcal{M}^{\otimes n} \otimes \mathbb{1}_{anc})(\Phi)) \right\}$$

Prepare input state: p is population of state, γ is a coherence term

$$\rho \equiv \begin{pmatrix} 1-p & \gamma^* \\ \gamma & p \end{pmatrix}$$

$$|\gamma| \leqslant \sqrt{(1-p)p}$$

$$p \in [0,1]$$

$$\mathcal{D}_\eta(\rho) = \begin{pmatrix} 1-\eta p & \sqrt{\eta}\,\gamma^* \\ \sqrt{\eta}\,\gamma & \eta p \end{pmatrix}$$

$$\lambda_\pm(\eta) \equiv \left(1 \pm \sqrt{(1-2\,\eta\,p)^2 + 4\,\eta\,|\gamma|^2}\right)/2$$

Quantum Capacity (cont'd)

- Need to maximize the coherent information, to do this take γ = 0.

$$S(\mathcal{D}_\eta(\rho)) = H_2(\lambda_+(\eta))$$

$$S((\mathcal{D}_\eta \otimes \mathbb{1}_{anc})(\Phi)) \;=\; H_2(\lambda_+(1-\eta))$$

$$J(p, |\gamma|^2) \equiv H_2(\lambda_+(\eta)) - H_2(\lambda_+(1-\eta))$$

$$Q \;\equiv\; \max_{p\in[0,1]} \left\{ H_2(\eta\, p) - H_2((1-\eta)\, p) \right\}$$

Entanglement Assisted Classical Capacity

$$C_E \equiv \max_{\rho \in \mathcal{H}} \left\{ S(\rho) + S(\mathcal{M}(\rho)) - S((\mathcal{M} \otimes \mathbb{1}_{anc})(\Phi)) \right\}$$

Using the same inputs as with the quantum capacity, we simply add the input entropy of the message to the coherent information we derived earlier to maximize the quantum mutual information.

$$I(p, |\gamma|^2) \equiv J(p, |\gamma|^2) + H_2\left(\frac{1 + \sqrt{(1-2p)^2 + 4|\gamma|^2}}{2}\right)$$

$$C_E \equiv \max_{p \in [0,1]} \left\{ H_2(p) + H_2(\eta\, p) - H_2((1-\eta)\, p) \right\}$$

Classical Capacity

- Compute C1 a lower bound for classical capacity.

$$C_n \equiv \max_{\xi_k, \rho_k \in \mathcal{H}^{\otimes n}} \left\{ S(\mathcal{M}^{\otimes n}(\rho)) - \sum_k \xi_k S(\mathcal{M}^{\otimes n}(\rho_k)) \right\}$$

$$\rho \equiv \sum_k \xi_k \rho_k$$

$$\rho_k \equiv \begin{pmatrix} 1 - p_k & \gamma_k^* \\ \gamma_k & p_k \end{pmatrix}$$

$$\chi \equiv H_2\left(\frac{1 + \sqrt{(1 - 2\,\eta\,p)^2 + 4\,\eta\,|\gamma|^2}}{2}\right)$$

$$-\sum_k \xi_k H_2\left(\frac{1 + \sqrt{(1 - 2\,\eta\,p_k)^2 + 4\,\eta\,|\gamma_k|^2}}{2}\right)$$

$$p = \sum_k \xi_k p_k \qquad \gamma \equiv \sum_k \xi_k \gamma_k$$

Classical Capacity (cont'd)

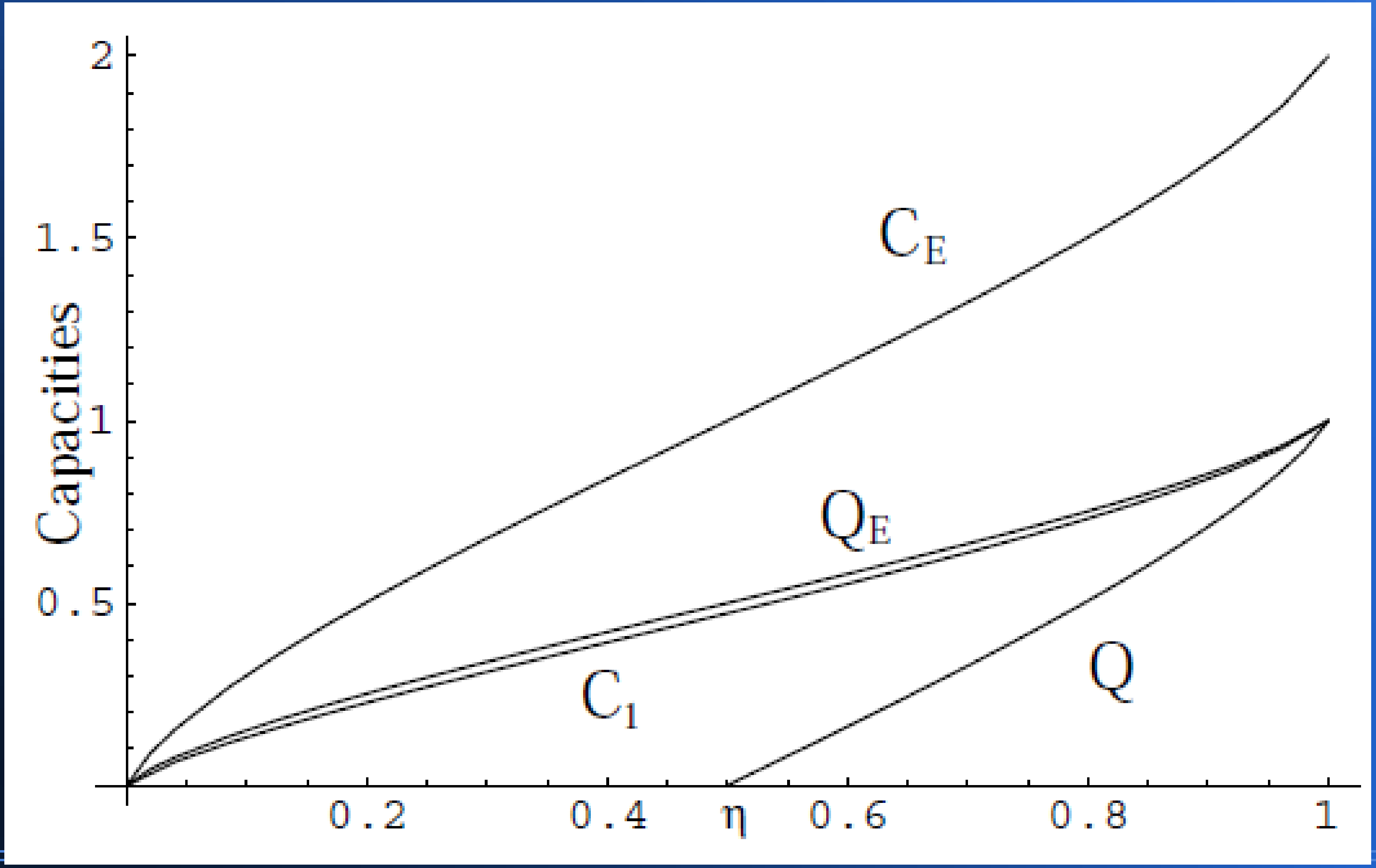
- First, we will find an upper bound, then find values that satisfy this bound

$$\sum_k \xi_k H_2\left(\frac{1+\sqrt{(1-2\,\eta\, p_k)^2+4\,\eta\,|\gamma_k|^2}}{2}\right) \geqslant \sum_k \xi_k H_2\left(\frac{1+\sqrt{1-4\,\eta\,(1-\eta)p_k^2}}{2}\right) \geqslant H_2\left(\frac{1+\sqrt{1-4\,\eta\,(1-\eta)(\sum_k \xi_k p_k)^2}}{2}\right)$$

- Upper bound: $C_1 \leqslant \max_{p\in[0,1]} \left\{H_2\,(\eta\, p) - H_2\left(\frac{1+\sqrt{1-4\,\eta\,(1-\eta)\,p^2}}{2}\right)\right\}$

- Satisfied by: $\xi_k = 1/d \qquad p_k = p$
  $\gamma_k = e^{2\pi i k/d}\,\sqrt{(1-p)p}$

Numerical Analysis of Capacities



Effectiveness of Channel

- Efficiency drops as $|r-s|^{(-2/3)}$
- Raises issues for long distance quantum communication
- Various possibilities for future study to improve such a channel's ability to transmit over notable distance.

Implications

- Spin-chain described by an amplitude damping channel
- Defined a method whereby the capacities can be investigated
- Found the capacities for the channel