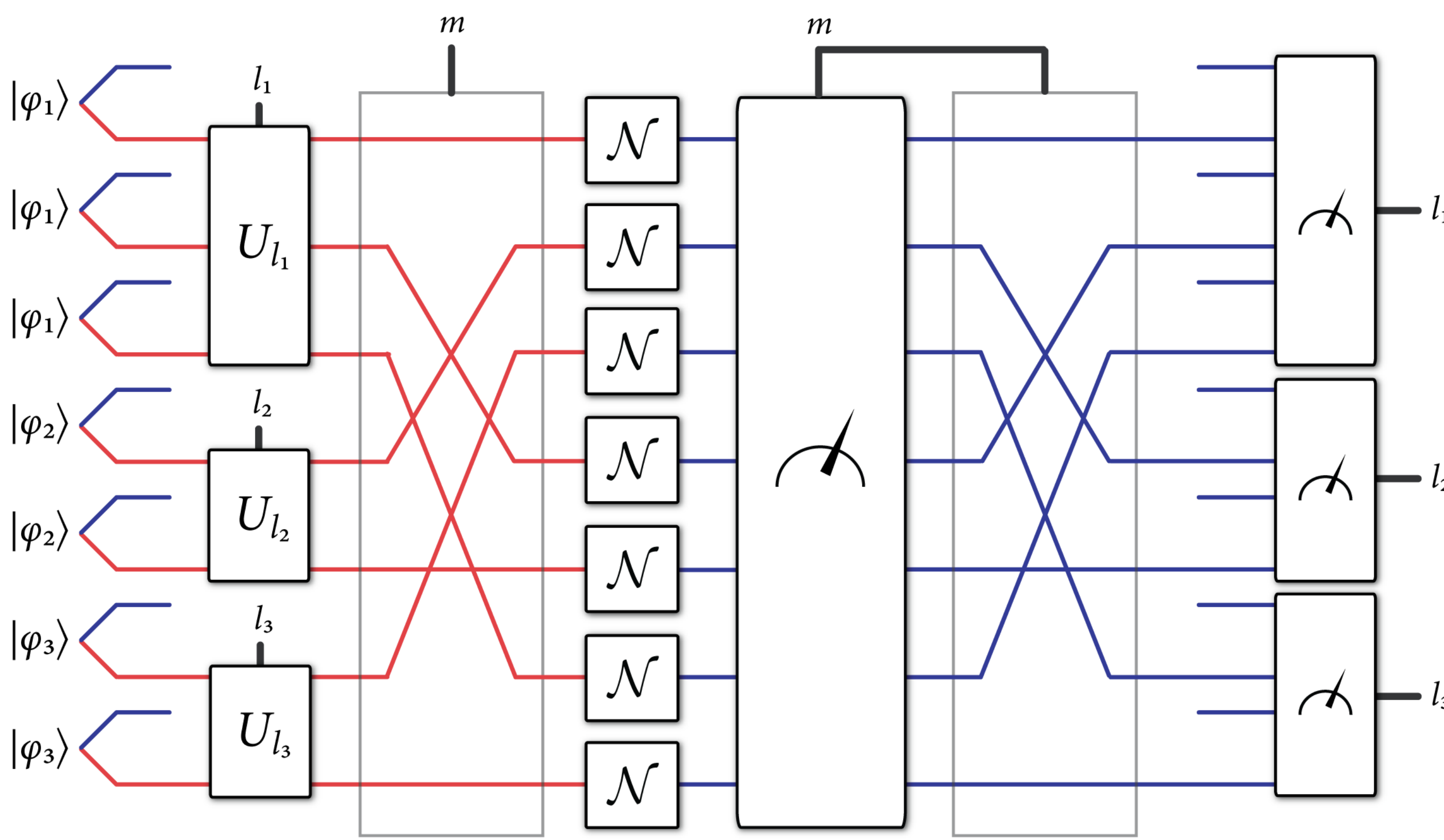
TRADE-OFF CODING



Presentation by Nan YANG

Starting Point

Alice wishes to send classical data to Bob over a quantum channel

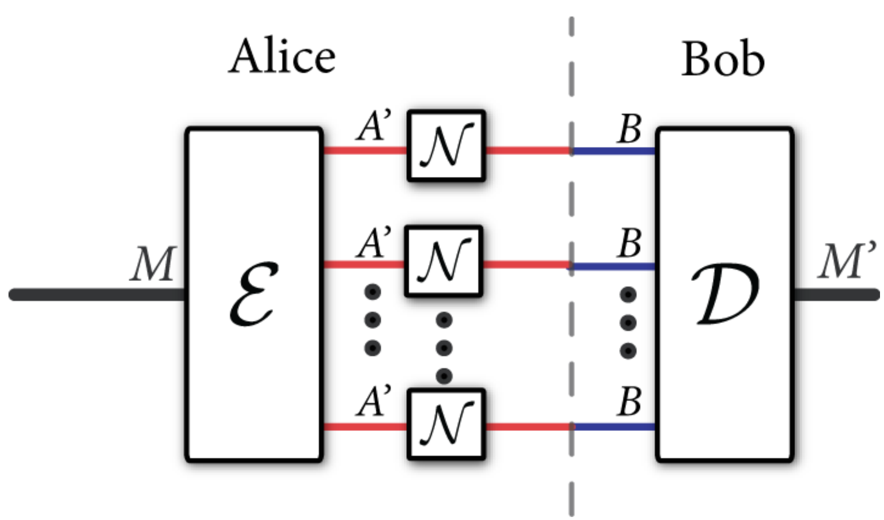
Share unlimited entanglement?

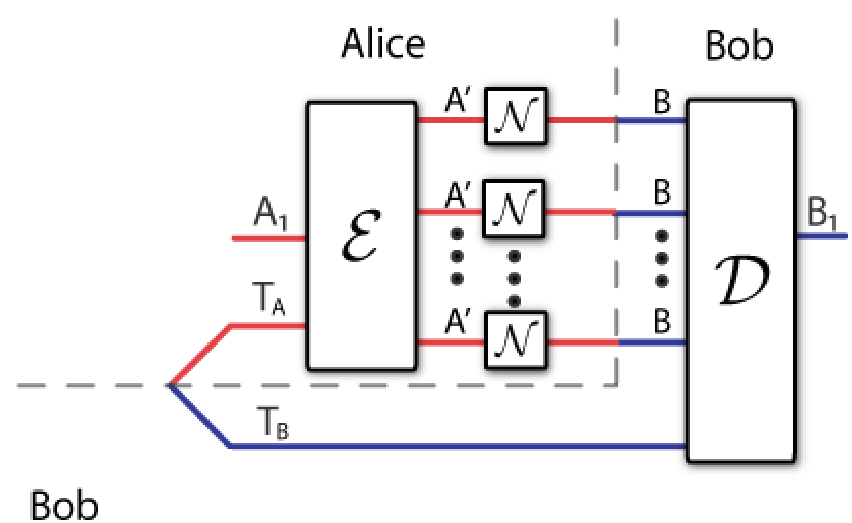
NO

Can achieve classical capacity equal to the Holevo information of the quantum channel $\chi(N)$

YES

Can achieve entanglement assisted classical capacity $I(N)$

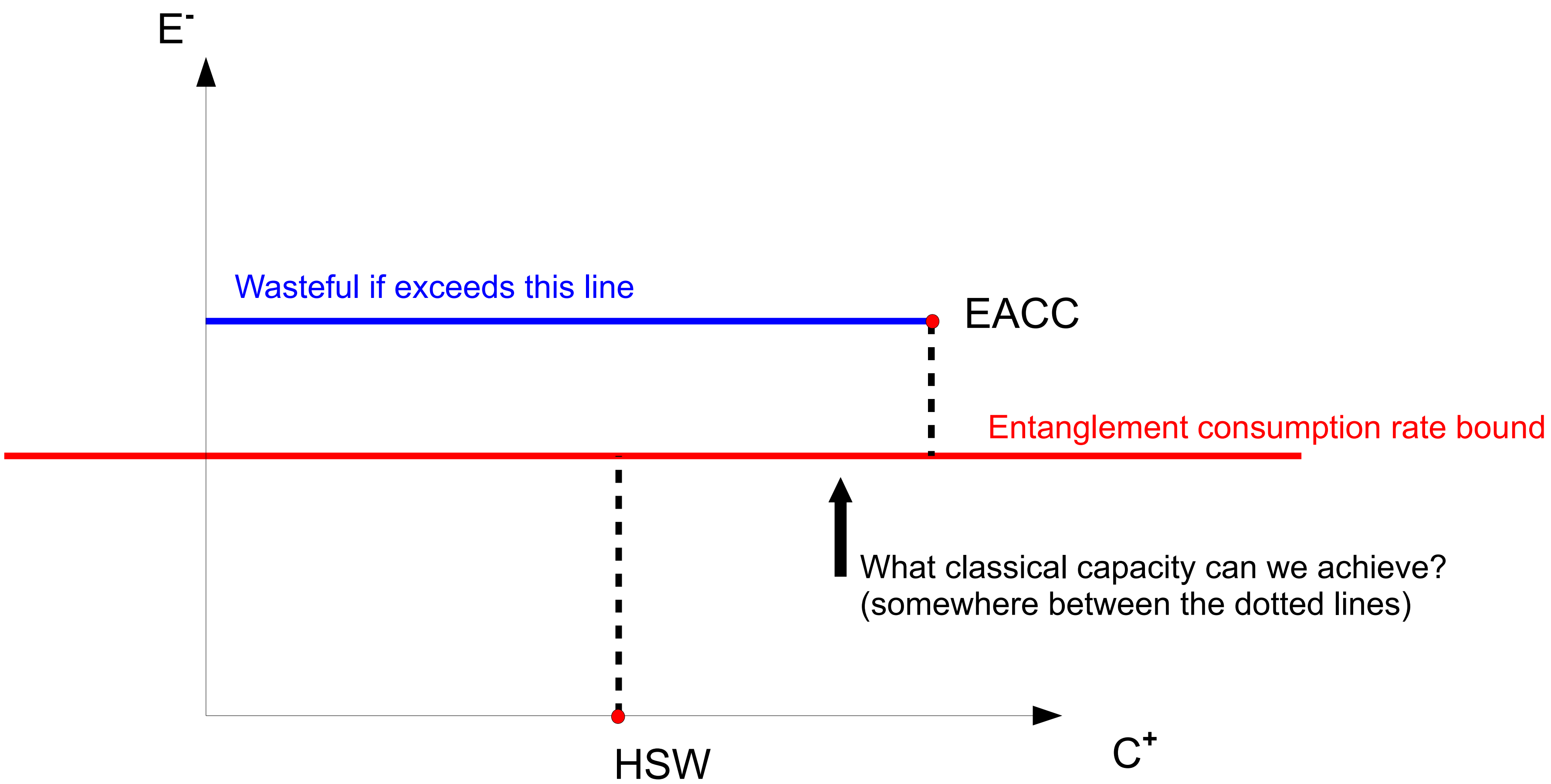




Entanglement Consumption

In the real world, entanglement is expensive

Would like get the most bang for our buck: maximize classical capacity given bounded entanglement consumption rate

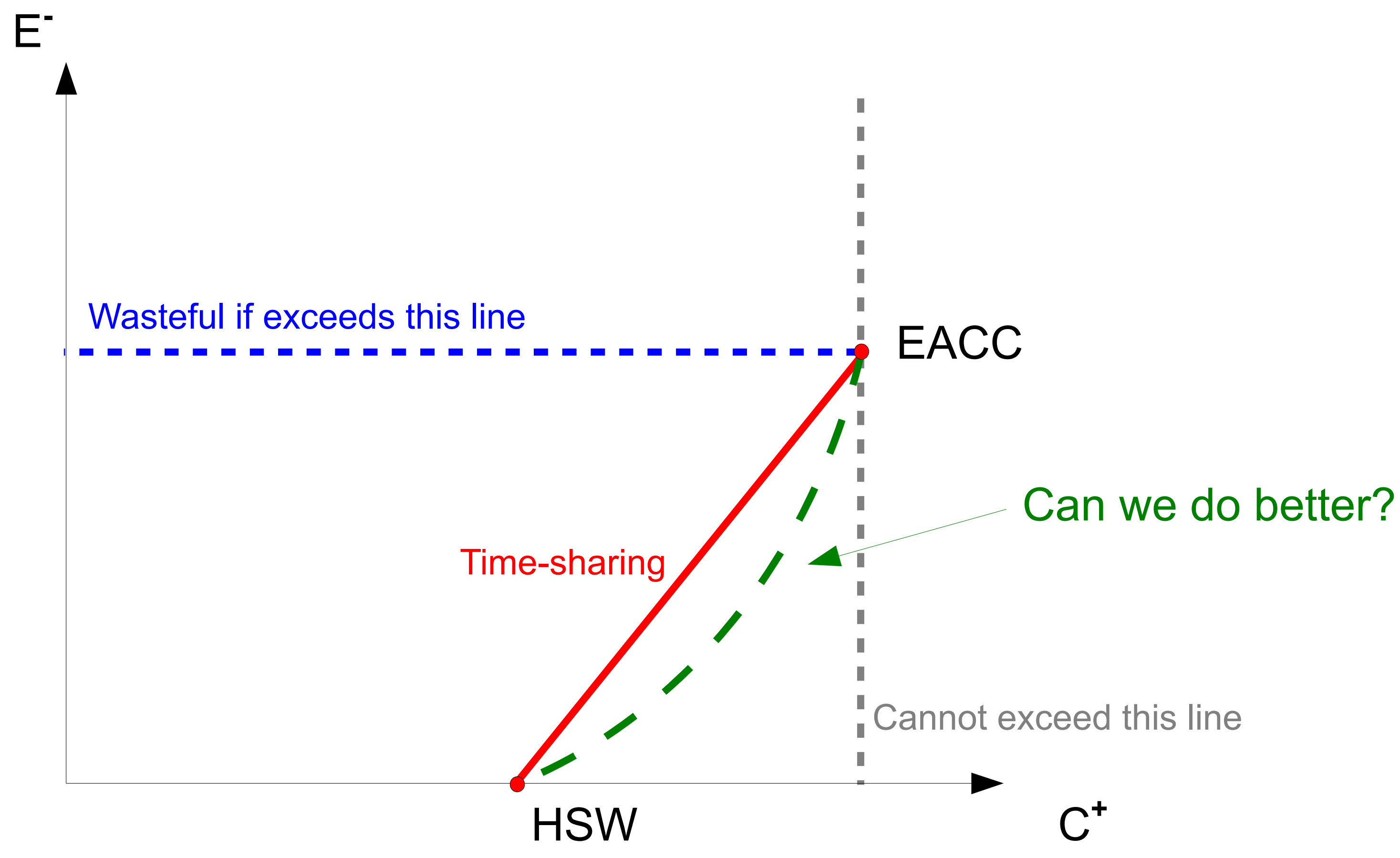


Time-sharing

Alternate between HSW and EACC according to fixed proportion $0 \leq \lambda \leq 1$

Can achieve rate of $\lambda \chi(N) + (1-\lambda) I(N)$

Corresponds to the straight line between HSW and EACC



YES!* Trade-Off Coding

*sometimes, depending on the channel

Alice and Bob have an HSW codebook $\left\{\rho_{x^n(m)}\right\}_m$

Each codeword is strongly typical, so each character appears (for simplicity) exactly some number of times. There is therefore a "proto" codeword R corresponding to a lexicographically re-ordering of any codeword:

$$R = \underbrace{\rho_{a_1} \otimes \cdots \otimes \rho_{a_1}}_{np_X(a_1)} \otimes \underbrace{\rho_{a_2} \otimes \cdots \otimes \rho_{a_2}}_{np_X(a_2)} \otimes \cdots \otimes \underbrace{\rho_{a_{|\mathcal{X}|}} \otimes \cdots \otimes \rho_{a_{|\mathcal{X}|}}}_{np_X(a_{|\mathcal{X}|})}$$

Any codeword differs from R by a permutation:

$$\pi\_m(R) = \rho_{x^n(m)} = \rho_{x_1(m)} \otimes \rho_{x_2(m)} \otimes \cdots \otimes \rho_{x_n(m)}$$

Purify R, and assume that Bob holds the purification system. This is their shared entanglement.

$$\underbrace{\varphi_{a_1} \otimes \cdots \otimes \varphi_{a_1}}_{np_X(a_1)} \otimes \underbrace{\varphi_{a_2} \otimes \cdots \otimes \varphi_{a_2}}_{np_X(a_2)} \otimes \cdots \otimes \underbrace{\varphi_{a_{|\mathcal{X}|}} \otimes \cdots \otimes \varphi_{a_{|\mathcal{X}|}}}_{np_X(a_{|\mathcal{X}|})}$$

Back to slide 1 for handwaving

Trade-Off Coding Rates

Alice and Bob have the following state after the protocol

$$\rho^{XAB} \equiv \sum_x p_X(x) |x\rangle\langle x|^X \otimes \mathcal{N}^{A'\to B}(\varphi_x^{AA'})$$

Chain rule for quantum mutual information implies that

$$I(AX;B)_\rho = I(X;B)_\rho + I(A;B|X)_\rho$$

They achieve $I(X;B)_\rho$ because Bob decodes the HSW codeword

Decoding each block gives

$$\frac{\#\text{ of bits generated}}{\#\text{ of channel uses}} \approx \frac{\sum_x n\, p_X(x)\, I(A;B)_{\rho_x}}{\sum_x n\, p_X(x)} = \sum_x p_X(x)\, I(A;B)_{\rho_x} = I(A;B|X)_\rho .$$

Amount of entanglement consumed is

$$\frac{\#\text{ of ebit consumed}}{\#\text{ of channel uses}} \approx \frac{\sum_x n\, p_X(x)\, H(A)_{\rho_x}}{\sum_x n\, p_X(x)} = \sum_x p_X(x)\, H(A)_{\rho_x} = H(A|X)_\rho .$$

Giving us the resource inequality

$$\langle \mathcal{N} \rangle + H(A|X)_\rho [qq] \geq I(AX;B)_\rho [c \to c]$$

Time-sharing a special case of Trade-Off Coding

$$\sigma^{UXAB} \equiv (1-\lambda)\,|0\rangle\langle 0|^{U} \otimes |0\rangle\langle 0|^{X} \otimes \mathcal{N}^{A'\to B}(\phi^{AA'}) + \lambda\,|1\rangle\langle 1|^{U} \otimes \sum_x p_X(x)\,|x\rangle\langle x|^{X} \otimes |0\rangle\langle 0|^{A} \otimes \mathcal{N}^{A'\to B}(\psi_x^{A'})$$

Classical information communicated with trade-off code is

$$\begin{aligned} I(AUX;B)_\sigma &= I(A;B|XU)_\sigma + I(X;B|U)_\sigma + I(U;B)_\sigma \\ &= (1-\lambda)\,I(A;B)_{\mathcal{N}(\phi)} + \sout{\lambda\left[\sum_x p_X(x)\,I(A;B)_{|0\rangle\langle 0|\otimes\mathcal{N}(\psi_x)}\right]} + \\ &\quad \sout{(1-\lambda)\,I(X;B)}_{|0\rangle\langle 0|\otimes\mathcal{N}(\phi)} + \lambda I(X;B)_{\{p(x),\psi_x\}} + I(U;B)_\sigma \\ &\geq (1-\lambda)\,I(\mathcal{N}) + \lambda\,\chi(\mathcal{N})\,. \end{aligned}$$

Thus trade-off coding reduces to time sharing for channels such that $I(U;B)_\sigma = 0$

Dynamic Capacity

$$\sigma^{XAB} \equiv \sum_x p(x) |x\rangle \langle x|^X \otimes \mathcal{N}^{A' \to B}(\phi_x^{AA'})$$

One-shot, one-state region $\mathcal{C}^{(1)}_{\mathrm{CQE},\sigma}(\mathcal{N})$ are those rate triples (C,Q,E) such that

$$C + 2Q \leq I(AX;B)_\sigma,$$
$$Q + E \leq I(A\rangle BX)_\sigma,$$
$$C + Q + E \leq I(X;B)_\sigma + I(A\rangle BX)_\sigma$$

The dynamic capacity region of quantum channel is given by

$$\mathcal{C}_{\mathrm{CQE}}(\mathcal{N}) = \overline{\bigcup_{k=1}^{\infty} \frac{1}{k} \bigcup_\sigma \mathcal{C}^{(1)}_{\mathrm{CQE},\sigma}(\mathcal{N}^{\otimes k})}$$

Hsieh and Wilde proved that a rate triple for a channel is achievable if and only if it lies within the dynamic capacity region.

What we did previously is a 2-D slice of this region.

Dynamic Capacity (cont'd)

Classically-enhanced father protocol achieves the quantum dynamic capacity region.

$$\begin{bmatrix} C \\ Q \\ E \end{bmatrix} = \begin{bmatrix} 0 & 2 & -2 \\ -1 & -1 & 1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} + \begin{bmatrix} I(X;B)_\sigma \\ \frac{1}{2} I(A;B|X)_\sigma \\ -\frac{1}{2} I(A;E|X)_\sigma \end{bmatrix}$$

A little algebra shows that this implies the inequalities from previous slide.

The converse, that any coding cannot do better than the dynamic capacity region, was proven by Hsieh and Wilde directly using

- Alicki-Fannes' inequality
- chain rule for quantum information
- the quantum data processing inequality

Dynamic Capacity Formula

The quantum dynamic capacity formula of a channel N is given by

$$D_{\lambda,\mu}\left(\mathcal{N}\right) \equiv \max_{\sigma} I\left(AX;B\right)_{\sigma} + \lambda I\left(A\rangle BX\right)_{\sigma} + \mu\left(I\left(X;B\right)_{\sigma} + I\left(A\rangle BX\right)_{\sigma}\right)$$

Its regularized version is given by

$$D_{\lambda,\mu}^{reg}\left(\mathcal{N}\right) \equiv \lim_{n\to\infty} \frac{1}{n} D_{\lambda,\mu}\left(\mathcal{N}^{\otimes n}\right)$$

If it is additive for a channel, then

$$D_{\lambda,\mu}^{reg}\left(\mathcal{N}\right) = D_{\lambda,\mu}\left(\mathcal{N}\right)$$

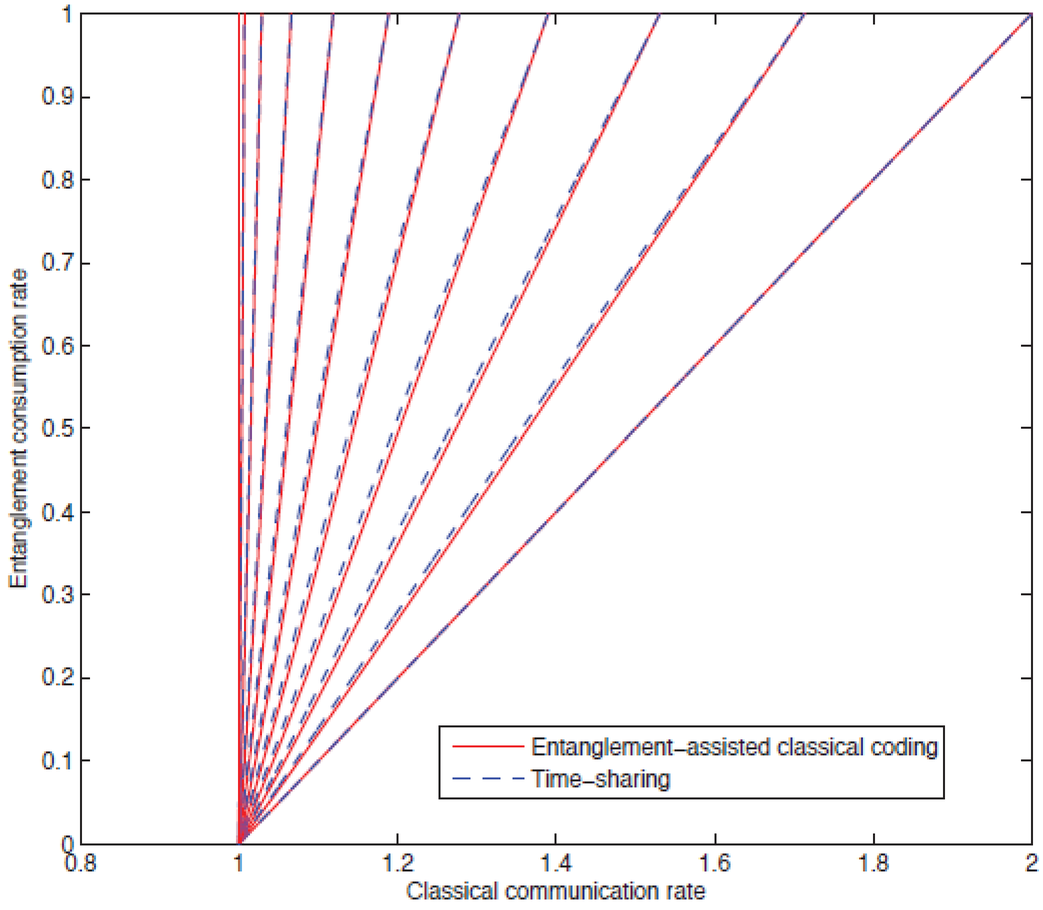
In which case the dynamic capacity region single-letterizes

$$\mathcal{C}_{\mathrm{CQE}}(\mathcal{N}) = \overline{\bigcup_{k=1}^{\infty} \frac{1}{k} \bigcup_{\sigma} \mathcal{C}_{\mathrm{CQE},\sigma}^{(1)}(\mathcal{N}^{\otimes k})} = \mathcal{C}_{\mathrm{CQE}}^{(1)}(\mathcal{N})$$

And computation of its boundary points becomes tractable. So far only Hadamard channels and the erasure channel are known to have single-letter dynamic capacity regions.

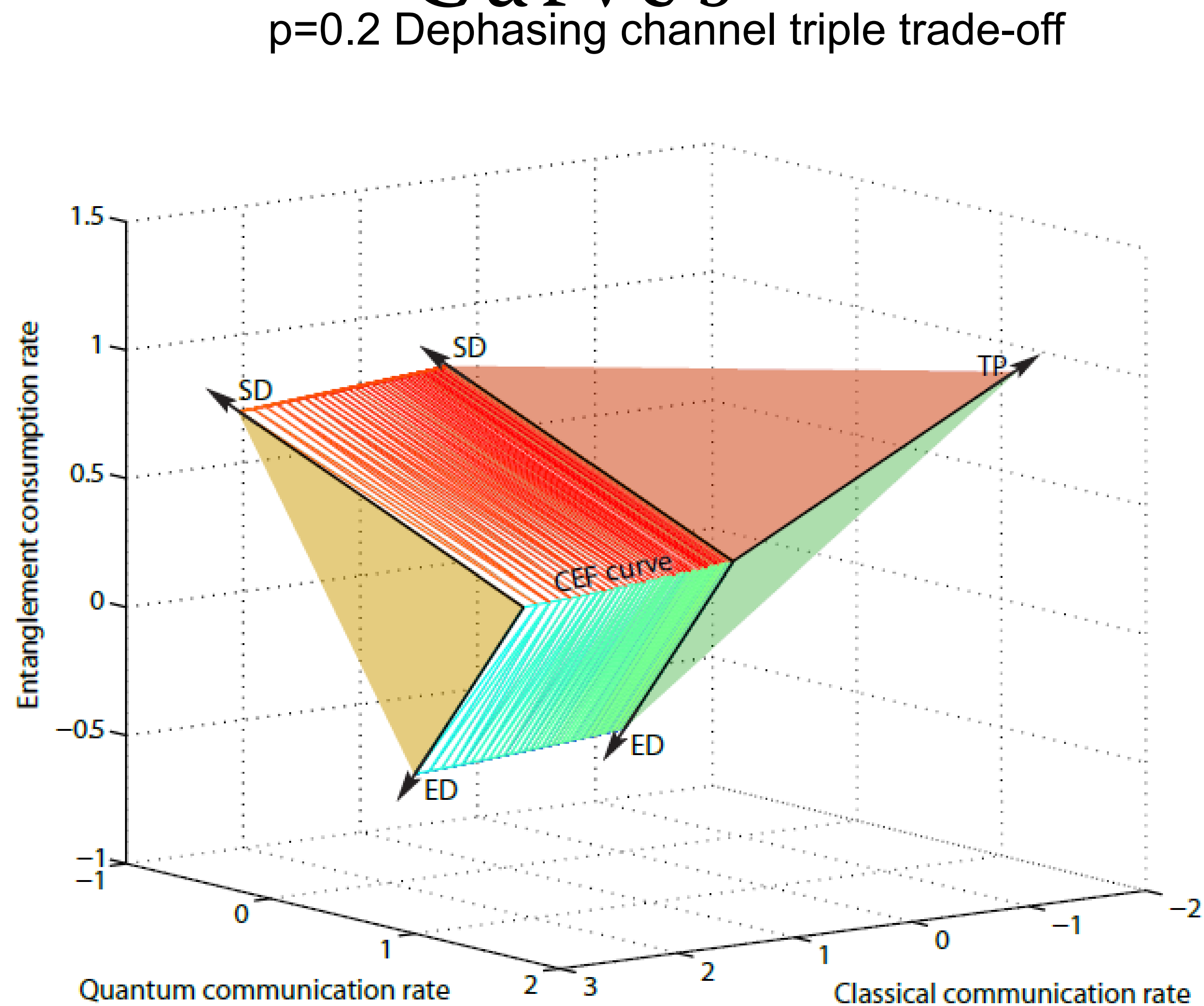
Examples of Channels and Trade-Off Curves

p-Dephasing channel classical-entanglement trade-off



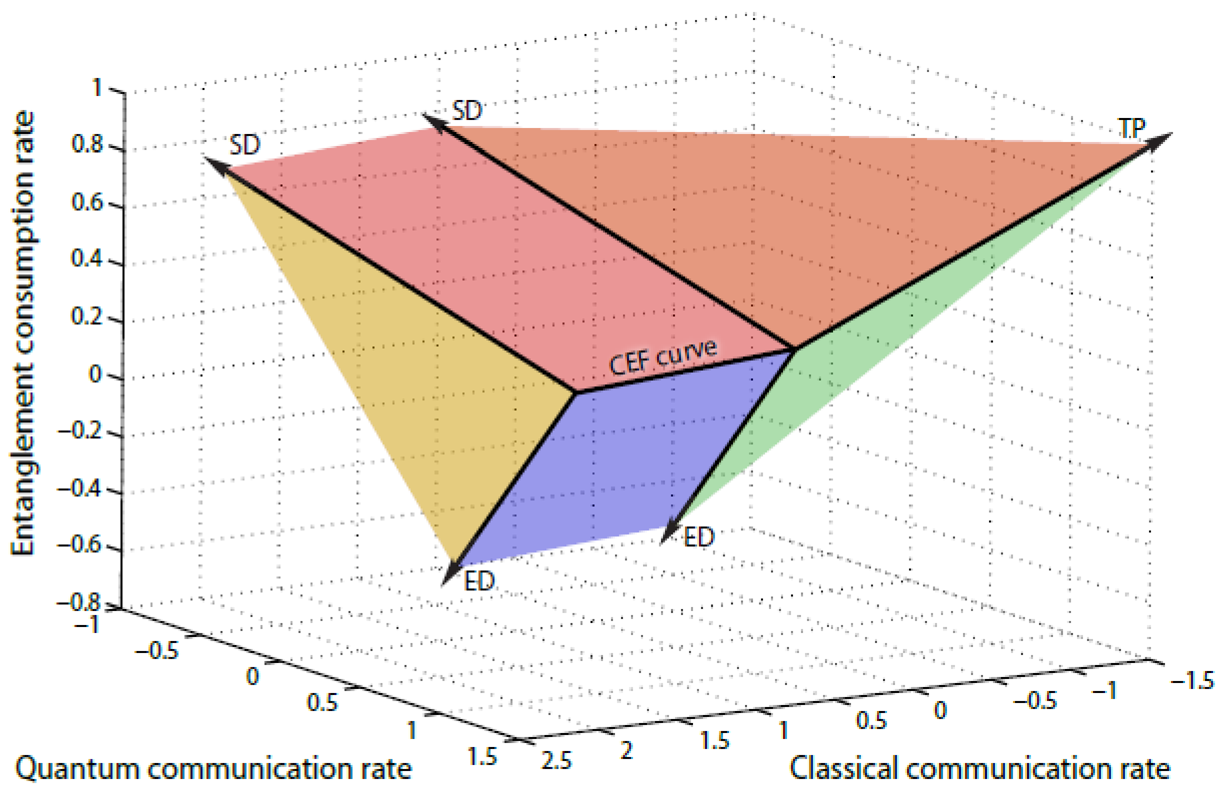
Counter-clockwise p=0, 0.1, 0.2, ...,1

Examples of Channels and Trade-Off Curves



Examples of Channels and Trade-Off Curves

P=0.25 erasure channel triple trade off



Conclusion

- Rate triple achievable if and only if it's in the quantum dynamic capacity region of that channel

- Need deeper understanding of why trade-off beats time-sharing for some channels

- Other channels (besides Hadamard and erasure) for which the dynamic capacity region single-letterizes?

- Does the dynamic capacity region correspond to some physical law?