On the quantum, classical and total amount of correlations in a quantum state

Berry Groisman, Sandu Popescu, and Andreas Winter

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History (forward and backwards)

- Classical - Landauer’s thermodynamic principle: amount of information stored is equal to amount of work necessary to erase it.

- Previous work - Oppenheim and $H^3$ showed that the work achievable by two parties on a bipartite state is less than work on the whole state.

- Schumacher’s “entropy exchange” will be the quantity of choice to measure the work in a quantum state.

- Calculated the amount necessary to erase all correlations in a quantum state, as well as quantum and classical correlations individually.

- Conjecture: quantum correlations $\leq$ classical correlations for any bipartite state.

- Operational interpretation (and straightforward proof) of strong subadditivity of mutual information.
A simple example

\[\{0,1\}_{u.r.}\]

\[|\Phi^+\rangle\]

\[\text{spherical state}\]

\[\text{No correlations left}\]

\[\text{spherical state}\]
A simple example

Begin with the state →

\[ |\Phi^+\rangle = \frac{1}{\sqrt{2}} \left( |00\rangle^{AB} + |11\rangle^{AB} \right). \]

Apply \( \mathbb{I}^{AB} \) or \( Z^A \otimes \mathbb{I}^B \) with equal probability →

\[ \rho = \frac{1}{2} |00\rangle\langle 00|^{AB} + \frac{1}{2} |11\rangle\langle 11|^{AB}. \]

Apply \( \mathbb{I}^{AB} \) or \( X^A \otimes \mathbb{I}^B \) with equal probability →

\[ \rho' = \pi^A \otimes \pi^B. \]

Thus 2 bits are required to erase the total correlations in the state (1 classical, 1 pure entanglement).
Definition (Randomizing map)

Let \( R \) be as follows:

\[
R : \rho^{AB} \rightarrow \sum_{i=1}^{N} p_i \left( U_i^A \otimes V_i^B \right) \rho \left( U_i^A \otimes V_i^B \right)^\dagger
\]

\( R \) \( \varepsilon \)-decorrelates a state \( \rho^{AB} \) if there exists \( \omega^A \otimes \omega^B \) such that

\[
\left\| R(\rho) - \omega^A \otimes \omega^B \right\|_1 \leq \varepsilon
\]

We call \( R \) a COLUR map. If all \( V_i = \mathbb{I} \) then it is a \( A \)-LUR map. If all \( U_i = \mathbb{I} \) then it is a \( B \)-LUR map. The composition of \( A \)-LUR and \( B \)-LUR is a LUR map.

Definition (Entropy exchange)

For a purification \( |\psi\rangle \langle \psi|^ZAB \) of \( \rho^{AB} \) and the map \( R \), we define

\[
S_e(R^{AB}, \rho^{AB}) := S \left( \left( \mathbb{I}^Z \otimes R^{AB} \right) |\psi\rangle \langle \psi| \right)
\]
First note that
\[ \log N \geq H(p) \geq S_e(R, \rho). \]

**Lemma (Size of COLUR maps)**

Any \( \varepsilon \) – decorrelating COLUR map \( R \) on \( A^n B^n \) has the lower bound
\[ S_e(R, \rho^\otimes n) \geq n (I(A; B) - O(\varepsilon)) \]

The above can be proved entirely via concavity of entropy and Fannes inequality

**Lemma (Size of A-LUR maps)**

There exists an \( \varepsilon \) – decorrelating A-LUR map \( R \) on \( A^n B^n \) with the upper bound
\[ \log N \leq n (I(A; B) + O(\varepsilon)) \]
Total bipartite correlations

Typical projection

COLUR on smaller subspace
Lemma (Chernoff bound)

Let $X_1, ..., X_N$ be i.i.d. random variables taking values in the operator interval $[0; \mathbb{I}]$ and with expectation $\mathbb{E}X \geq \mu \mathbb{I}$. Then for $0 \leq \varepsilon \leq 1$,

$$\Pr \left\{ \frac{1}{N} \sum_i X_i \notin [(1 - \varepsilon)\mathbb{E}X; (1 - \varepsilon)\mathbb{E}X] \right\} \leq \exp \left( -N \frac{\mu \varepsilon^2}{2} \right).$$

Our random variable is $U_i$ and $N = 2^n(I(A;B) + 4\varepsilon)$ is sufficient to make this bound $\leq 1$. 

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Together, the two lemmas give an extremely robust statement

**Theorem**

The amount of local noise needed to turn $\rho^{AB}$ into a product state is measured by

$$\sup_{\varepsilon} \lim_{n \to \infty} \inf \frac{1}{n} \min \left\{ S_\varepsilon (R, \rho \otimes^n) : R, \varepsilon \in \text{COLUR} \right\}$$

$$= \sup_{\varepsilon} \lim_{n \to \infty} \inf \frac{1}{n} \min \left\{ \log N : R, \varepsilon \in \text{A-LUR} \right\}$$

$$= I(A;B)$$

“Smallest $R$ for worst $\varepsilon$ in the asymptotic limit”
Implementing the correlated unitary randomizing map requires providing $i$ to Alice and Bob via the state

$$\gamma = \sum_i p_i |i\rangle^A \otimes |i\rangle^B$$

We consider the least cost of erasing $\rho \otimes \gamma$ minus the cost of erasing $\gamma$. We know this to be $I(A; B)$ by the theorem. However, this implies catalysis does not increase the cost.
Subadditivity

Clearly the green process simulates the red process $\implies I(A : BC) \geq I(A : B)$
Quantum correlations

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![Diagram showing the relationship between total correlations, classical correlations, and separable states in a quantum state.](image)
We define two quantities, the first is well-motivated, the second is desirable,

\[
Cl_{er}(\rho) := \sup_{\varepsilon > 0} \limsup_{n \to \infty} \sup_{\|\sigma - R(\rho \otimes^n)\|_1 \leq \varepsilon} \frac{1}{n} I(A; B)_{\sigma}
\]

\[
Cl^*_{er}(\rho) := \sup_{\varepsilon > 0} \limsup_{n \to \infty} \sup_{\|\sigma - T(\rho \otimes^n)\|_1 \leq \varepsilon} \frac{1}{n} I(A; B)_{\sigma}
\]

where \( T \) is any local CPTP map.

**Remark** For pure states it is shown that \( E_{er}(\psi) = Cl^*_{er}(\psi) = \frac{1}{2} C_{er}(\psi) = E(\psi) \). However for mixed states the optimal paths of erasure do not necessarily coincide.
Open questions

- Do the cost of entanglement erasure ($E_{er}$) and optimistic cost ($\overline{E}_{er}$) coincide asymptotically?

$$\sup_{\varepsilon > 0} \lim_{n \to \infty} \inf_{\|\sigma - R(\rho \otimes n)\|_1 \leq \varepsilon} \frac{1}{n} S(\sigma) - S(\rho) = \lim_{n \to \infty} \inf_{\sigma = R(\rho \otimes n)} \frac{1}{n} S(\sigma) - S(\rho)$$

- Is $E_{er}$ monotonic under local operations and classical communication? Is it convex?

- Is $E_{er}$ a bound on the distillable entanglement and is $C_{er}$ a bound on the distillable secret key?