The Capacity of the Quantum Depolarizing Channel

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Background

- The depolarizing channel

$$\Delta_\lambda(\rho) = \lambda\rho + \frac{1-\lambda}{d} I$$

- Minimum output entropy

$$H_{\min}(\Psi) = \inf_\rho H(\Psi(\rho))$$

- p-norm and maximal p-norm

$$\|A\|_p = \left(\mathrm{Tr}\left\{A^p\right\}\right)^{1/p} \qquad v_p(\Psi) = \sup_\rho \|\Psi(\rho)\|_p$$

Background

It is known that

- $$\frac{d}{dp}\|\rho\|_p \Bigg|_{p=1} = -H(\rho)$$

  and hence

  $$\frac{d}{dp}v_p(\Psi) \Bigg|_{p=1} = -H_{\min}(\Psi)$$

- The additivity of the minimum output entropy is equivalent to the additivity of the Holevo information.

Background

- Let $\mathcal{B} = \{|\psi_i\rangle\}$ be an orthonormal basis, and let $E_i = |\psi_i\rangle\langle\psi_i|$. The dephasing channel corresponding to $\mathcal{B}$ is

$$\Phi_\lambda(\rho) = \lambda\rho + (1-\lambda)\sum_{i=1}^{d} E_i \rho E_i$$

- We say a vector is **uniform** if all entries have the same absolute value.

Main Results

- **Theorem 1** The classical capacity of the quantum depolarizing channel Δ_λ is $\chi(\Delta_\lambda)$.
- **Theorem 2** For any channel Ψ

$$\chi(\Delta_\lambda \otimes \Psi) = \chi(\Delta_\lambda) + \chi(\Psi)$$

- **Theorem 3** For any channel Ψ, and any $p \geq 1$

$$v_p(\Delta_\lambda \otimes \Psi) = v_p(\Delta_\lambda) v_p(\Psi)$$

and hence

$$H_{\min}(\Delta_\lambda \otimes \Psi) = H_{\min}(\Delta_\lambda) + H_{\min}(\Psi)$$

Three Lemmas

- **Lemma 1** For any state τ^{AB}, let U be any unitary matrix, and $\tau'^{AB} = (U \otimes I)\tau^{AB}(U^\dagger \otimes I)$.

$$\|(\Delta_\lambda \otimes \Psi)(\tau^{AB})\|_p = \|(\Delta_\lambda \otimes \Psi)(\tau'^{AB})\|_p$$

- **Lemma 2** Convex decomposition of the depolarizing channel

$$\Delta_\lambda(\rho) = \sum_{n=1}^{2d^2(d+1)} c_n U_n^\dagger \Phi_\lambda^{(n)}(\rho) U_n$$

Three Lemmas

- **Lemma 3** For any state τ^{AB}, define

$$\tau_{(i)}^{B} = \mathrm{Tr}_A \left\{ (E_i \otimes I)\tau^{AB} \right\}$$

Then

$$\|(\Phi_\lambda \otimes I)(\tau^{AB})\|_p \leq d^{(1-1/p)} v_p(\Delta_\lambda) \left(\sum_{i=1}^{d} \mathrm{Tr} \left\{ \tau_{(i)}^{B} \right\}^p \right)^{1/p}$$

Proof of Theorem 3

- $$v_p(\Delta_\lambda \otimes \Psi) = v_p(\Delta_\lambda) v_p(\Psi)$$

- By Lemma 2, it's sufficient to show

$$\| \left(\Phi_\lambda^{(n)} \otimes \Psi \right) (\rho^{AB}) \|_p \leq v_p(\Delta_\lambda) v_p(\Psi)$$

- Apply Lemma 3 with

$$\tau^{AB} = (I \otimes \Psi)(\rho^{AB})$$

$$\tau_{(i)}^{B} = \Psi \left(\rho_{(2)}^{B} \right) = \Psi \left(\mathrm{Tr}_A \left\{ (E_i \otimes I) \rho^{AB} \right\} \right)$$

The Convex Decomposition

- Intermediate channel

$$\Omega_\lambda(\rho) = \Delta_\lambda(\rho) + \frac{1-\lambda}{d}\left(\rho - \mathrm{diag}(\rho)\right)$$

- Define diagonal matrix G with $G_{kk} = \exp\left(\frac{2\pi i k}{d}\right)$.
- **Lemma 4**

$$\Delta_\lambda(\rho) = \frac{\lambda d}{1+(d-1)\lambda}\Omega_\lambda(\rho) + \frac{1-\lambda}{1+(d-1)\lambda}\frac{1}{d}\sum_{k=1}^{d}(G^\dagger)^k \Omega_\lambda(\rho) G^k$$

The Convex Decomposition

- Define diagonal matrix H with $H_{kk} = \exp\left(\frac{2\pi i k^2}{2d^2}\right)$.
- $|\theta\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d} |i\rangle$.
- For $k = 1, \ldots, d$, and $a = 1, \ldots, 2d^2$, define

$$\left|\psi_{k,a}\right\rangle = G^k H^a |\theta\rangle$$

$$E_{k,a} = \left|\psi_{k,a}\right\rangle \left\langle\psi_{k,a}\right|$$

The Convex Decomposition

- Define the dephasing channels

$$\Phi_\lambda^{(a)}(\rho) = \lambda\rho + (1-\lambda)\sum_{k=1}^{d} E_{k,a}\rho E_{k,a}$$

- **Lemma 5**

$$\Omega_\lambda(\rho) = \frac{1}{2d^2}\sum_{a=1}^{2d^2} \Phi_\lambda^{(a)}$$

- Sufficient to show that

$$\frac{1}{2d}\sum_{a=1}^{2d^2}\sum_{k=1}^{d} E_{k,a}\rho E_{k,a} = I + \rho - \mathrm{diag}(\rho)$$

The Dephasing Channel

- Only left to show Lemma 3, for all τ^{AB}, define

$$\tau_{(i)}^{B} = \mathrm{Tr}_A \left\{ (E_i \otimes I)\tau^{AB} \right\}$$

Then

$$\|(\Phi_\lambda \otimes I)(\tau^{AB})\|_p \leq d^{(1-1/p)} v_p(\Delta_\lambda) \left(\sum_{i=1}^{d} \mathrm{Tr} \left\{ \tau_{(i)}^{B} \right\}^p \right)^{1/p}$$

- Calculations. . .

The Dephasing Channel

- Input state defined on $\mathbb{C}^{d\times d} \otimes \mathbb{C}^{d'\times d'}$.
- Rewrite

$$\rho^{AB} = \left(\sqrt{\rho^{AB}}\right)^{\dagger} \sqrt{\rho^{AB}}$$

$$\sqrt{\rho^{AB}} = (V_1, \ldots, V_d)$$

  Where V_i is a $dd' \times d'$ matrix.
- Calculations. . .

Conclusion and Future Work

- Proof of a long-conjectured property for depolarizing channel.
- Depolarizing channel as a convex combination of simpler channels.
- Future work: Finding explicit constructions of channels that violate additivity.