COMP 598 Winter 2011
Homework 2

Due Thursday 10 February 2011.

First part: Exercises 4.1.5, 4.1.6, 4.1.7, 4.1.8, 4.1.11, 4.2.1, 4.2.2, 4.3.3, 4.3.4, 4.3.6, 4.4.6 in *From Classical Information Theory to Quantum Shannon Theory*

Second part: The below exercises

1. Are the following operators valid density operators? For (a-d), give their representation on the Bloch sphere if they are valid density operators.

   (a)
   $$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

   (b)
   $$\begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{2}{3} \end{bmatrix}$$

   (c)
   $$\begin{bmatrix} 3/4 & 1/2 \\ 1/2 & 1/4 \end{bmatrix}$$

   (d)
   $$\begin{bmatrix} 1/2 & (1+i)/4 \\ (1-i)/4 & 1/2 \end{bmatrix}$$

   (e) Let $\{|n\rangle\}_{n=0}^{\infty}$ be some orthonormal basis. For $N \geq 0$, define the following operator. Is it a density operator?
   $$\frac{1}{N+1}\sum_{n=0}^{\infty}\left(\frac{N}{N+1}\right)^{n}|n\rangle\langle n|$$

2. Suppose that σ is a density operator with the following spectral decomposition:
   $$\sigma^{A} = \sum_{i} p(i)\,|i\rangle\langle i|^{A}.$$
   Let $|\psi\rangle^{RA}$ be the following purification of σ:
   $$|\psi\rangle^{RA} = \sum_{i}\sqrt{p(i)}\,|i\rangle^{R}|i\rangle^{A}.$$
   Show that
   $$|\psi\rangle^{RA} = \sqrt{d}\left(I^{R}\otimes\sqrt{\sigma}^{A}\right)|\Phi\rangle^{RA},$$
   where $|\Phi\rangle^{RA}$ is the maximally entangled qudit state:
   $$|\Phi\rangle^{RA} = \frac{1}{\sqrt{d}}\sum_{i}|i\rangle^{R}|i\rangle^{A}.$$

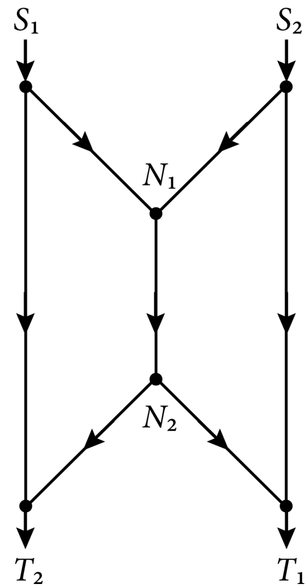

Figure 1: The "bitrapezoidal" network.

3. You will explore a quantum network communication protocol that is a "quantization" of a classical scheme. First, consider the "bitrapezoidal" communication network depicted in Figure 1. The labeling indicates that there are two source nodes S_1 and S_2, two intermediate routing nodes N_1 and N_2, and two destination or target nodes T_1 and T_2. The directed edges between the nodes indicate that there is a classical noiseless bit channel between these nodes.

   (a) The transmitting party at S_1 would like to communicate one classical bit to the target node T_1, and the transmitting party at S_2 would like to communicate one classical bit to the target node T_2. They would like to do so with as few uses of the network as possible. One naive strategy is for S_1 to the send her bit in the order $S_1 \rightarrow N_1 \rightarrow N_2 \rightarrow T_1$ and for S_2 to send her bit in the order $S_2 \rightarrow N_1 \rightarrow N_2 \rightarrow T_2$. Unfortunately, this strategy requires two uses of the network because it requires two uses of the channel from N_1 to N_2. Show that it is possible for S_1 and S_2 to transmit their bits by using the network just once and by exploiting a network coding strategy (Hint: consider allowing for the XOR of incoming bits at some of the nodes).

   (b) We will now determine how to "quantize" the above classical strategy. Suppose now that the network is the same as it is above, with the only difference being that each channel connecting two nodes is now a noiseless qubit channel. Such a quantization is not possible directly, given that the classical strategy involves copying bits and the no-cloning theorem prevents us from copying qubits. So, we will allow one bit of forward and backward classical communication (in addition to the noiseless quantum communication) between each connected node in the network and determine a quantization of the classical strategy. We will replace each copy node in the network (an example is in Figure 2 (a)) with the circuit

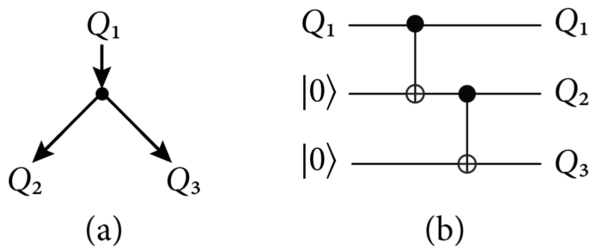

Figure 2: Quantization of classical copy.

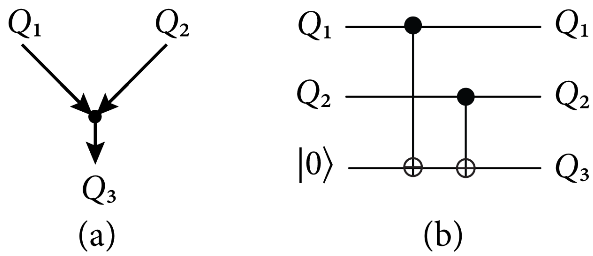

Figure 3: Quantization of a classical XOR.

given in Figure 2 (b). Determine the action of the quantized version of the copy node on an arbitrary qubit state $\sum_{z\in\{0,1\}} \beta_z |z\rangle$.

(c) We will replace each XOR node in the network (an example is in Figure 3 (a)) with the quantization given in Figure 3 (b). Determine the action of the quantized version of the XOR node on an arbitrary two-qubit state $\sum_{z_1,z_2} \beta_{z_1,z_2} |z_1\rangle |z_2\rangle$.

(d) What is the final state resulting from a node-by-node simulation of the classical protocol? (Hint: you should have a state that lives on 11 qubit registers—two of them are S_1 and S_2, two of them are T_1 and T_2, and the seven others are "internal registers" kept at certain nodes in the network. If the initial state is $\sum_{x,y\in\{0,1\}} \alpha_{x,y} |x\rangle^{S_1} |y\rangle^{S_2}$, the final state has the bits x and y (from the initial basis states) spread across 11 qubit registers.) It is helpful to draw out the quantum circuit acting on the 11 registers as a quantum circuit diagram.

(e) The final state that you have obtained is an entangled state on 11 registers, when in fact, our goal is to have the final state be $\sum_{x,y\in\{0,1\}} \alpha_{x,y} |x\rangle^{T_1} |y\rangle^{T_2}$ on the target registers. We now need to determine how to disentangle all of the other registers so that the final state on T_1 and T_2 is the aforementioned state. Consider the circuit in Figure 4. Suppose that the state on registers Q_1 and Q_2 is as follows:

$$\sum_{z\in\{0,1\}} \beta_z |z\rangle^{Q_1} |z\rangle^{Q_2}.$$

Show that the action of a Hadamard on the second qubit, measurement of the second qubit in the computational basis, and conditional application of a Z gate on the first qubit (depending on the measurement outcome) effectively disentangles

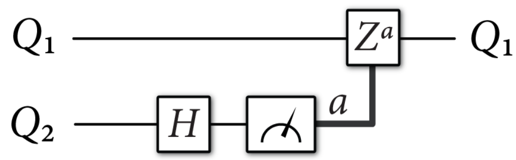

Figure 4: Disentangling circuit.

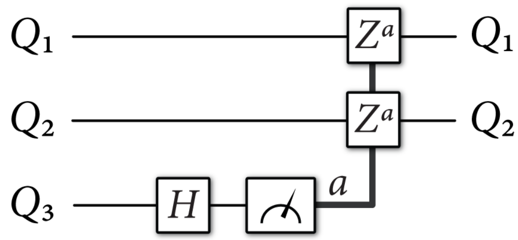

Figure 5: Another disentangling circuit.

the registers Q_1 and Q_2 so that the state becomes

$$\left(\sum_{z \in \{0,1\}} \beta_z \left| z \right\rangle^{Q_1} \right) \left| x \right\rangle^{Q_2},$$

where $x \in \{0, 1\}$ (the value of x does not matter—it just matters that the registers Q_1 and Q_2 are disentangled).

(f) We also need a way to disentangle three registers in a state of the following form:

$$\sum_{x,y} \gamma_{x,y} \left| x \right\rangle^{Q_1} \left| y \right\rangle^{Q_2} \left| x \oplus y \right\rangle^{Q_3}.$$

Show that the output of the circuit in Figure 5 is the following state:

$$\left(\sum_{x,y} \gamma_{x,y} \left| x \right\rangle^{Q_1} \left| y \right\rangle^{Q_2} \right) \left| z \right\rangle^{Q_3},$$

where $z \in \{0, 1\}$.

(g) Use these disentangling circuits to disentangle the state on 11 registers. The final state should be of the following form:

$$\sum_{x,y \in \{0,1\}} \alpha_{x,y} \left| x \right\rangle^{S_1} \left| y \right\rangle^{S_2} \left| x \right\rangle^{T_1} \left| y \right\rangle^{T_2}. \tag{1}$$

(Hint: Perform the disentangling circuits in the reverse order of the network, so that you remove the internal registers that you created in part (d). You should feedback the phase corrections so that they apply to the registers from which others were created. This step should only require one bit of backward classical communication per edge in the network.)

(h) Show how to disentangle the state in (1) by exploiting the disentangling circuits. The final state should be of the following form:

$$\sum_{x,y\in\{0,1\}} \alpha_{x,y} |x\rangle^{T_1} |y\rangle^{T_2},$$

and you should only require one bit of forward classical communication per edge in the network. (Hint: Consider exploiting the result from part (a).)