Lecture 22

21 APR 2014

We have now proved that

Local Hamiltonian (LH) $\in$ QMA

& YES instances of any QMA promise problem can be mapped to YES instances of LH. We did this via the Feynman-Kitaev circuit-to-Hamiltonian construction. We will now finish off the proof that <u>5-LH is QMA-hard</u> by mapping NO instances of any QMA promise problem to NO instances of 5-LH. We will then be able to conclude that 5-LH is QMA-complete, thus finishing off the proof of the "quantum Cook-Levin theorem."

To map NO instances of QMA to NO instances of LH, we need to find a lower bound on the minimum eigenvalue of our constructed Hamiltonian:

$$H_{in} + H_{out} + H_{prop}$$

It is true that H_{in} & H_{out} commute, but these do not necessarily commute w/ H_{prop} (b/c H_{prop} has the full quantum computation encoded in it). So we will require something nontrivial.

This is where "Kitaev's geometrical lemma" comes into play:

Let $A_1, A_2 \geq 0$, such that the minimum non-zero eigenvalue of both operators is lower bounded by $v > 0$. Suppose that the null spaces L_1 & L_2 of A_1 & A_2 have trivial intersection $L_1 \cap L_2 = \{0\}$. Then

$$A_1 + A_2 \geq 2v \cdot \sin^2\left(\frac{\alpha(L_1, L_2)}{2}\right) I$$

where the angle between two subspaces X & Y

is defined as

$$\alpha(\mathcal{X},\mathcal{Y}) \quad \text{such that}$$

$$\cos(\alpha(\mathcal{X},\mathcal{Y})) \equiv \max_{\substack{|x\rangle\in\mathcal{X},\\ |y\rangle\in\mathcal{Y}}} |\langle x|y\rangle|$$

where maximization is over unit vectors in each space.

Proof: From the definition of v, we have that

$$A_1 \geq v(I-\Pi_{L_1}) \;\&\; A_2 \geq v(I-\Pi_{L_2})$$

where Π_{L_i} is the projector onto the space L_i for $i\in\{1,2\}$. So it then suffices to prove that

$$v(I-\Pi_{L_1}) + v(I-\Pi_{L_2}) \geq 2v\sin^2\left(\frac{\alpha(L_1,L_2)}{2}\right) I$$

This is equivalent to

$$2vI - 2v\sin^2\left(\frac{\alpha(L_1,L_2)}{2}\right) I \geq v(\Pi_{L_1}+\Pi_{L_2})$$

which is the same as

$$2\left[1-\sin^2\left(\frac{\alpha(L_1,L_2)}{2}\right)\right] I \geq \Pi_{L_1}+\Pi_{L_2}$$

Using trig. identity $\cos(2\theta)=1-2\sin^2(\theta)$, this is equivalent to

$$\left[1+\cos(\alpha(L_1,L_2))\right] I \geq \Pi_{L_1}+\Pi_{L_2}$$

so we will focus on proving this one...

So we need to upper bound all the eigenvalues of $\Pi_{L_1}+\Pi_{L_2}$. To this end, suppose that $|\varphi\rangle$ is an eigenvector w/ eigenvalue λ, i.e., $(\Pi_{L_1}+\Pi_{L_2})|\varphi\rangle = \lambda|\varphi\rangle$

Then let $|x_1\rangle$ & $|x_2\rangle$ be unit vectors & u_1 & u_2 real & non-negative such that

$$\Pi_{L_1}|\varphi\rangle = u_1|x_1\rangle \quad \& \quad \Pi_{L_2}|\varphi\rangle = u_2|x_2\rangle$$

Since

$$\langle\varphi|\Pi_{L_1}|\varphi\rangle = (\langle\varphi|\Pi_{L_1})(\Pi_{L_1}|\varphi\rangle)$$
$$= (\langle x_1|u_1)(u_1|x_1\rangle)$$
$$= u_1^2$$

this means ~~that~~

$$\lambda = \langle\varphi|(\Pi_{L_1}+\Pi_{L_2})|\varphi\rangle = u_1^2+u_2^2$$

But we also know that

$$\lambda|\varphi\rangle = (\Pi_{L_1}+\Pi_{L_2})|\varphi\rangle$$
$$= u_1|x_1\rangle + u_2|x_2\rangle$$

implying that

$$\lambda^2 = [\langle\varphi|\lambda][\lambda|\varphi\rangle] = (\langle x_1|u_1+\langle x_2|u_2)(u_1|x_1\rangle + u_2|x_2\rangle)$$
$$= u_1^2+u_2^2+2u_1u_2\,\mathrm{RE}\{\langle x_1|x_2\rangle\}$$

Let $\gamma = \mathrm{RE}\{\langle x_1 | x_2 \rangle\}$

Putting these together, we get that

$$(1+|\gamma|)\delta - \delta^2 = (1+|\gamma|)(u_1^2+u_2^2) - (u_1^2+u_2^2+2u_1u_2\gamma)$$

$$= u_1^2|\gamma| + u_2^2|\gamma| - 2u_1u_2\gamma$$

$$= |\gamma|(u_1^2+u_2^2 \pm 2u_1u_2)$$

$$= |\gamma|(u_1 \pm u_2)^2$$

$$\geq 0$$

So we have $(1+|\mathrm{RE}\{\langle x_1|x_2\rangle\}|)\delta \geq \delta^2$

which means that

$$\delta \leq 1+|\mathrm{RE}\{\langle x_1|x_2\rangle\}|$$

$$\leq 1+\cos(\alpha(L_1,L_2))$$

which proves the lemma. $\square$

So we now need to figure out how to apply the lemma to our case.

We will take $A_1 = H_{in} + H_{out}$ &

$$A_2 = H_{prop}$$

So we need to figure out v & $\cos(\measuredangle(L_1, L_2))$ for this choice.

Consider that A_1 is a sum of commuting projectors. Since smallest non-zero eigenvalue of any projector is 1, this serves as a lower bound on λ_{min} for A_1 (a sum of 2 projectors).

Last time we argued that H_{prop} is partially diagonalized in such a way that the matrix acting on the clock register is tridiagonal, of the form

$$\begin{bmatrix} 1 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & -1 & \ddots & \end{bmatrix}$$

The eigenvalues are then

$$\lambda_k = 2\left(1 - \cos\left(\frac{\pi k}{L+1}\right)\right) \quad \text{where} \quad k \in \{0, \ldots, L\}$$

$\lambda_0 = 0$ so next largest non-zero eigenvalue is

$$\lambda_1 = 2\left(1 - \cos\left(\frac{\pi}{L+1}\right)\right) \geq c/L^2$$

for some positive constant c.

This follows from a Taylor series expansion for cosine.

So all of this means that the minimum non-zero eigenvalue of both A_1 & A_2 is bounded from below by $\frac{c}{L^2}$

(for L large enough)

We now need to reason about the angle between the null spaces of A_1 & A_2.

For this purpose, it makes things easier to apply $W^{\dagger}(\cdot)W$ to every term in the Hamiltonian where $W = \sum_{t=0}^{L} U_t \cdots U_1 \otimes |t\rangle\langle t|$ (recall that this uncomputes the Hamiltonian)

So then we can think of $\mathcal{L}_1$ decomposing as

$$\mathcal{L}_1 = \left[\mathcal{H}_P^{\otimes p(n)} \otimes |0\rangle_A^{\otimes n} \otimes |0\rangle_C \right] \oplus$$

$$\left[\left(\mathcal{H}^{\otimes p(n)+n}\right)_{PA} \otimes \operatorname{span}\{|1\rangle, \dots, |L-1\rangle\} \right]$$

$$\oplus$$

$$\left[U_1^\dagger \cdots U_L^\dagger \left(|1\rangle \otimes \mathcal{H}^{\otimes p(n)+n-1}\right) \otimes |L\rangle_C \right]$$

+

$$\mathcal{L}_2 = \left(\mathcal{H}^{\otimes p(n)+n}\right)_{PA} \otimes \frac{1}{\sqrt{L+1}} \sum_{t=0}^{L} |t\rangle$$

So we'll use this structure to estimate

$$\cos^2(\alpha(L_1, L_2))$$

We can write

$$\cos^2(\alpha(L_1, L_2)) = \max_{\substack{|x\rangle \in L_1, \\ |y\rangle \in L_2}} |\langle x|y\rangle|^2$$

$$= \max_{\substack{|x\rangle \in L_1, \\ |y\rangle \in L_2}} \langle y|x\rangle\langle x|y\rangle = \max_{|y\rangle \in L_2} \langle y|\Pi_{L_1}|y\rangle$$

(This last one is true b/c $\langle y|\Pi_{L_1}|y\rangle = \|\Pi_{L_1}|y\rangle\|_2^2$

$$= \max_{|x_1\rangle \in L_1} \langle x_1|\Pi_{L_1}|y\rangle$$

$$= \max |\langle x|y\rangle|^2 \Big)$$

To estimate this, we can use the structure (direct sum) of $\mathcal{L}_1$. Also, note that any $|y\rangle \in \mathcal{L}_2$ takes the form

$$|\psi\rangle \otimes \frac{1}{\sqrt{L+1}} \sum_{t=0}^{L} |t\rangle$$

So $\mathcal{L}_1$ breaks down into three orthogonal projections

For the second, it is

$$\langle y | \Pi_{K_2} | y \rangle = \frac{L-1}{L+1}$$ b/c every term in $\frac{1}{\sqrt{L+1}} \sum_{t=0}^{L} |t\rangle$ gives a contribution except for $|0\rangle$ & $|L\rangle$

For the 1^{st} & 3^{rd},

let $K_1 = H_P^{\otimes p(n)} \otimes |0\rangle_A^{\otimes n}$ &

$K_2 = U_1^{+} \ldots U_\Delta^{+} \left(|1\rangle \otimes H^{p(n)+n-1}\right)$

So the contributions from these are

$$\langle y | \left(\Pi_{K_1} \otimes |0\rangle\langle 0|_c + \Pi_{K_2} \otimes |L\rangle\langle L|_c \right) | y \rangle$$

$$= \frac{1}{L+1} \langle \psi | \Pi_{K_1} + \Pi_{K_2} | \psi \rangle$$

But we can use the lemma from before to find that

$$\frac{1}{L+1} \langle \varphi | \left(\Pi_{K_1} + \Pi_{K_2} \right) | \varphi \rangle \leq$$

$$\frac{1}{L+1} \left(1 + \cos^2(K_1, K_2) \right) \qquad (*)$$

But $\cos^2(K_1, K_2) = \max_{\substack{|k_1\rangle \in K_1, \\ |k_2\rangle \in K_2}} |\langle k_1 | k_2 \rangle|^2$

$$= \max_{|\psi\rangle} \left\| \langle 1 | \otimes I \, U_L \cdots U_1 |\psi\rangle_P |0\rangle_A \right\|_2^2$$

For NO instances, we have the promise that this is $\leq \varepsilon$

$$\Rightarrow (*) \leq \frac{1}{L+1} \left(1 + \sqrt{\varepsilon} \right)$$

Adding all three contributions gives that

$$\cos^2(\alpha(L_1, L_2)) \leq \frac{L-1}{L+1} + \frac{1 + \sqrt{\varepsilon}}{L+1}$$

$$= 1 - \frac{1 - \sqrt{\varepsilon}}{L+1}$$

$$\Rightarrow \sin^2(\alpha(L_1, L_2)) \geq \frac{1-\sqrt{\varepsilon}}{L+1}$$

can further use $\sin^2\left(\frac{x}{2}\right) \geq \frac{1}{4}\sin^2(x)$

to get

$$\sin^2\left(\frac{\alpha(L_1, L_2)}{2}\right) \geq \frac{1}{4}\sin^2(\alpha(L_1, L_2))$$

$$\geq \frac{1}{4}\left(\frac{1-\sqrt{\varepsilon}}{L+1}\right)$$

Putting all this together gives that in the case of a NO instance of QMA, we get that

$$\lambda_{min}(H_{in} + H_{out} + H_{prop}) \geq \frac{c}{L^2}\left(\frac{1}{4}\left(\frac{1-\sqrt{\varepsilon}}{L+1}\right)\right)$$

$$= \Omega\left(\frac{1-\sqrt{\varepsilon}}{L^3}\right)$$

and we're done...