Lecture 15

19 MAR 2014

Continue w/ Grover's algorithm...

The action of U_R on an arbitrary state is to perform an "inversion about the mean", i.e.,

for a state $\sum_x \alpha_x |x\rangle$

the mean amplitude is $\frac{1}{2^n} \sum_x \alpha_x = \mu$

So, we calculate

$$U_R \sum_x \alpha_x |x\rangle =$$

$$\left(2|+_n\rangle\langle +_n| - I\right) \sum_x \alpha_x |x\rangle$$

$$= \left(\frac{2}{2^n} \sum_{y,z} |y\rangle\langle z| - I\right)\left(\sum_x \alpha_x |x\rangle\right)$$

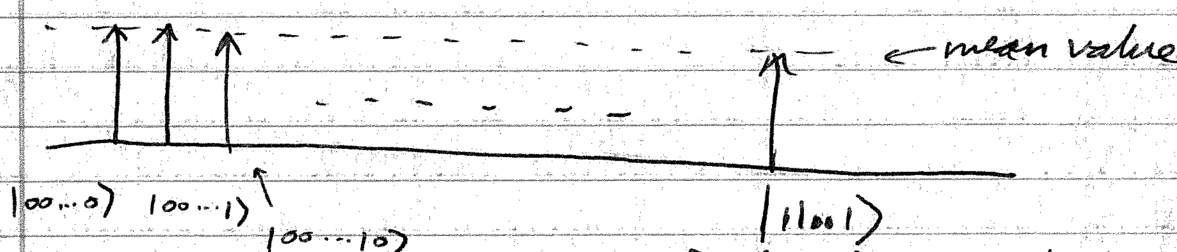
$$= \frac{2}{2^n} \sum_{y,z,x} \alpha_x |y\rangle\langle z|x\rangle - \sum_x \alpha_x |x\rangle$$

$$= \frac{2}{2^n} \sum_y \left(\sum_z \alpha_z\right) |y\rangle - \sum_x \alpha_x |x\rangle$$

$$= \sum_x (2\mu - \alpha_x) |x\rangle$$

Grover's algorithm only uses real amplitudes, so we can visualize it more easily.

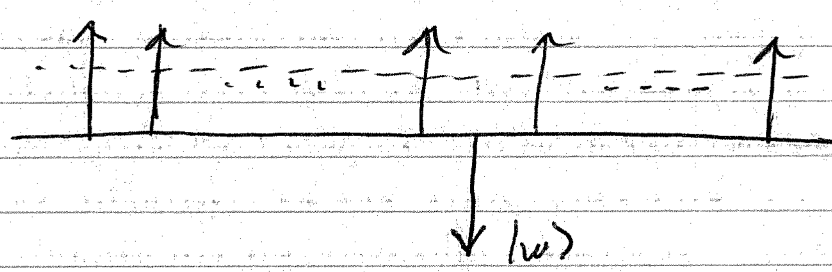
Initially the state is a uniform superposition looking like



amplitude for target is $\frac{1}{\sqrt{N}}$

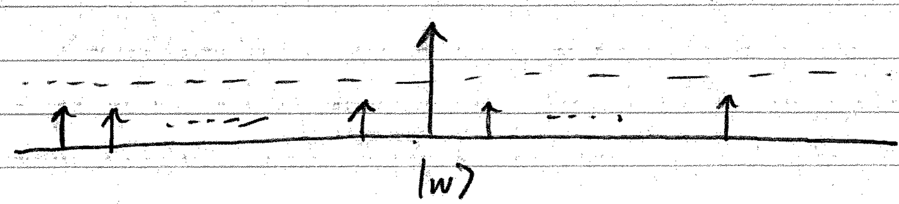
Now apply U_f

desired state will get inverted about 0.



so the mean value is lower.

Now apply U_R (inversion about the mean)



Amplitude of $|w\rangle$ increases by about $\frac{2}{\sqrt{N}}$ + all other amplitudes go down. The idea is to iterate this procedure about $O(\sqrt{N})$ times to have nearly all of the amplitude on the solution.

Keep repeating this ~~so~~ we add roughly $\frac{2}{\sqrt{N}}$ to the amplitude of $|w\rangle$ on each iteration.

For an analysis, it suffices to consider the 2D subspace spanned by

$$\{|w\rangle, |\psi_{bad}\rangle\}$$

The initial state is

$$|+_n\rangle = \frac{1}{\sqrt{N}}|w\rangle + \sqrt{\frac{N-1}{N}}\,|\psi_{bad}\rangle$$

Another basis for this subspace is

$$\{|+_n\rangle, |\overline{+_n}\rangle\}$$

where

$|\overline{+_n}\rangle$ is orthogonal to $|+_n\rangle$, i.e.,

$$|\overline{+_n}\rangle = \sqrt{\frac{N-1}{N}}\,|w\rangle - \frac{1}{\sqrt{N}}\,|+_{bad}\rangle$$

Let θ be such that

$$\sin\theta = \frac{1}{\sqrt{N}} \quad \&$$

$$\cos\theta = \sqrt{\frac{N-1}{N}}\,.$$

So

$$|+_n\rangle = \sin\theta\,|w\rangle + \cos\theta\,|+_{bad}\rangle$$

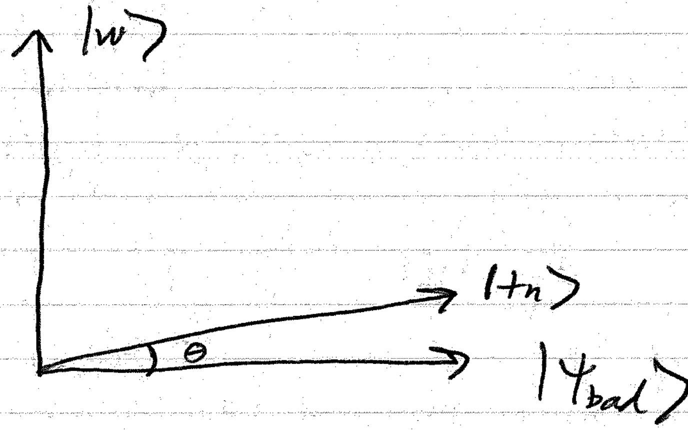
$$|\overline{+_n}\rangle = \cos\theta\,|w\rangle - \sin\theta\,|+_{bad}\rangle$$

Also

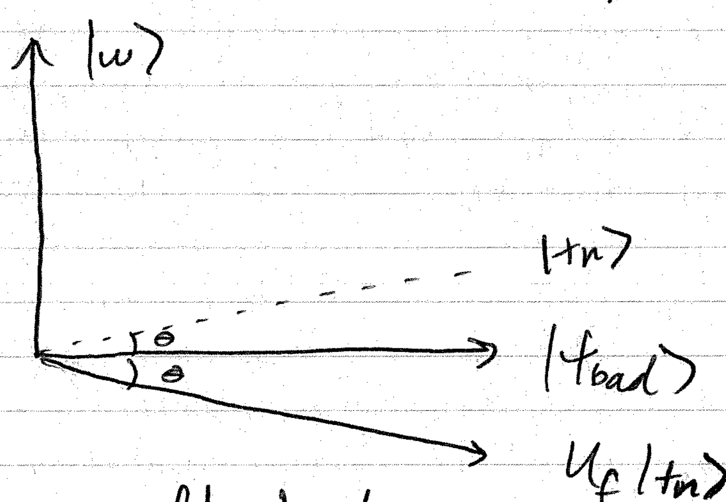
$$|w\rangle = \sin\theta\,|+_n\rangle + \cos\theta\,|\overline{+_n}\rangle$$

$$|+_{bad}\rangle = \cos\theta\,|+_n\rangle - \sin\theta\,|\overline{+_n}\rangle$$

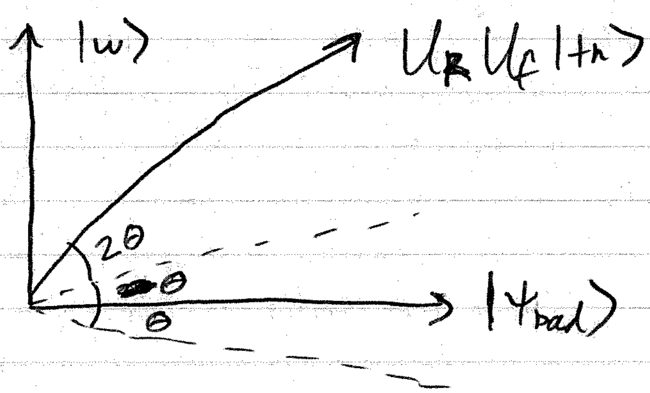
Initial picture:



U_f can be thought of as a reflection about $|\psi_{bad}\rangle$. This gives



Next, we reflect about $|t_n\rangle$, giving



So after the Grover iterate, ~~we~~ the state becomes

$$\sin(2\theta+\theta)\,|w\rangle + \cos(2\theta+\theta)\,|\psi_{bad}\rangle$$

Continuing this idea, after k Grover iterates, the state becomes

$$\sin((2k+1)\theta)\,|w\rangle + \cos((2k+1)\theta)\,|\psi_{bad}\rangle$$

So, in order to have a high probability of measuring $|w\rangle$ when performing a measurement of the 1st register, we would like to have

$$\sin((2k+1)\theta) \approx 1$$

which means that

$$(2k+1)\,\theta \approx \pi/2 \text{ \& thus}$$

$$k \approx \frac{\pi}{4\theta} - \frac{1}{2} \approx \frac{\pi}{4}\sqrt{N} = O(\sqrt{N})$$

(using $\sin\theta \simeq \theta$ for large N & $\sin\theta = \frac{1}{\sqrt{N}}$)