Lecture 10

4 MAR 2014

Last time discussed Deutsch-Jozsa & Simon's algorithm

We will discuss the phase estimation algorithm today.

To begin with, recall from Deutsch-Jozsa that the final step is to extract/decode information about relative phases by performing Hadamard gates.
It can also be used to encode information into relative phases, as in

$$H|x\rangle = \frac{1}{\sqrt{2}}\left[|0\rangle + (-1)^x|1\rangle\right]$$

$$= \frac{1}{\sqrt{2}} \sum_{y\in\{0,1\}} (-1)^{xy}|y\rangle$$

Of course by applying H to

$$\frac{1}{\sqrt{2}}\left[|0\rangle + (-1)^x|1\rangle\right]$$, we get

$|x\rangle$ & thus have decoded information in the relative phases.

As we discussed before, this holds for n-qubit states as well:

$$H^{\otimes n}|x\rangle = \frac{1}{\sqrt{2^n}} \sum_{y\in\{0,1\}^n} (-1)^{x\cdot y}|y\rangle$$

so that we encode into phases &

$$H^{\otimes n}\frac{1}{\sqrt{2^n}} \sum_{y\in\{0,1\}^n} (-1)^{x\cdot y}|y\rangle$$

$$= |x\rangle$$

so that we decode.

However the phases $(-1)^{x\cdot y}$ are of a very special form.

In general, a phase takes the form $e^{2\pi i \omega}$ where $\omega \in (0,1)$

so Hadamards cannot get more general phases

We need a generalization of the Hadamard gate to decode these more general phases. This is the quantum Fourier transform

Now suppose we are given a state

$$\frac{1}{\sqrt{2^n}} \sum_{y=0}^{2^n-1} e^{2\pi i w y} |y\rangle$$

should understand that sum is over integer values, but $|y\rangle$ is implicit for a binary representation of y.

Suppose that our goal is to estimate w. This is the phase estimation problem.

There is a well known quantum algorithm for this.

Given that $\omega \in (0,1)$, we can write ~~it~~ in binary as

$$\omega = 0.x_1 x_2 x_3 \cdots$$

where

$$0.x_1 x_2 x_3 \cdots = x_1 \cdot 2^{-1} + x_2 \cdot 2^{-2} + x_3 \cdot 2^{-3} + \cdots$$

Similarly, powers-of-2 multiples of ω:

$$2^k \omega = x_1 x_2 \cdots x_k . x_{k+1} x_{k+2} \cdots$$

Consider that

$$e^{2\pi i (2^k \omega)} = e^{2\pi i (x_1 x_2 \cdots x_k . x_{k+1} x_{k+2} \cdots)}$$

$$\overset{(1)}{=} e^{2\pi i (x_1 x_2 \cdots x_k)} e^{2\pi i (0.x_{k+1} x_{k+2} \cdots)}$$

$$\overset{(2)}{=} e^{2\pi i (0.x_{k+1} x_{k+2} \cdots)}$$

(2) b/c $e^{2\pi i l} = 1$ for integer l

(1) b/c $x_1 x_2 \cdots x_k . x_{k+1} x_{k+2} = x_1 x_2 \cdots x_k + 0.x_{k+1} x_{k+2} \cdots$

Elementary Phase Estimation

Suppose for simplicity that $\omega = 0.x_1$ & the input is a 1-qubit state.

Then the state is

$$\frac{1}{\sqrt{2}} \sum_{y \in \{0,1\}} e^{2\pi i (0.x_1) y} |y\rangle$$

$$= \frac{1}{\sqrt{2}} \sum_{y \in \{0,1\}} e^{2\pi i \left(\frac{x_1}{2}\right) y} |y\rangle$$

$$= \frac{1}{\sqrt{2}} \sum_{y \in \{0,1\}} e^{\pi i x_1 y} |y\rangle$$

$$= \frac{1}{\sqrt{2}} \sum_{y \in \{0,1\}} (-1)^{x_1 y} |y\rangle$$

& then we can recover x_1 by performing a Hadamard as before.

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Consider extending this identity to the case

$$\frac{1}{\sqrt{2^n}} \sum_{y=0}^{2^n-1} e^{2\pi i w y} |y\rangle$$

$$= \frac{1}{\sqrt{2^n}} \sum_{y^n \in \{0,1\}^n} e^{2\pi i w \left(\underbrace{\sum_{l=1}^{n} 2^{n-l} y_l}\right)} |y_1 \cdots y_n\rangle$$

rewriting $y = y_1 \cdots y_n$ in binary

$$= \frac{1}{\sqrt{2^n}} \sum_{y^n \in \{0,1\}^n} \prod_{l=1}^{n} e^{2\pi i w 2^{n-l} y_l} |y_1 \cdots y_n\rangle$$

$$= \frac{1}{\sqrt{2^n}} \sum_{y_1 \in \{0,1\}} \cdots \sum_{y_n \in \{0,1\}} e^{2\pi i w 2^{n-1} y_1} |y_1\rangle \cdots e^{2\pi i w 2^{0} y_n} |y_n\rangle$$

$$= \frac{1}{\sqrt{2^n}} \sum_{y_1 \in \{0,1\}} e^{2\pi i w 2^{n-1} y_1} |y_1\rangle \otimes \cdots \otimes \sum_{y_n \in \{0,1\}} e^{2\pi i w 2^{0} y_n} |y_n\rangle$$

(reasoning is that $\sum_{i,j} \alpha_i \beta_j |i\rangle|j\rangle = \sum_i \alpha_i |i\rangle \sum_j \beta_j |j\rangle$

$$= \left(\frac{|0\rangle + e^{2\pi i (0.x_n x_{n+1} \cdots)} |1\rangle}{\sqrt{2}}\right) \otimes \left(\frac{|0\rangle + e^{2\pi i (0.x_{n-1} x_n x_{n+1} \cdots)} |1\rangle}{\sqrt{2}}\right)$$

$$\otimes \cdots \otimes \left(\frac{|0\rangle + e^{2\pi i (0.x_1 x_2 \cdots)} |1\rangle}{\sqrt{2}}\right)$$

Now, for another example,

Suppose the state is on two qubits

$$\frac{1}{\sqrt{2^2}} \sum_{y=0}^{2^2-1} e^{2\pi i \omega y} |y\rangle$$

& suppose $\omega = 0.x_1x_2$

With the above identity, we get

$$\frac{1}{\sqrt{2^2}} \sum_{y=0}^{2^2-1} e^{2\pi i (0.x_1x_2)y} |y\rangle =$$

$$\left(\frac{|0\rangle + e^{2\pi i (0.x_2)}|1\rangle}{\sqrt{2}}\right) \otimes \left(\frac{|0\rangle + e^{2\pi i (0.x_1x_2)}|1\rangle}{\sqrt{2}}\right)$$

Observe that we can get x_2 by performing a Hadamard on the 1st qubit & measuring in the computational basis.

If $x_2 = 0$, then we just need to perform a Hadamard on the second qubit to get x_1.

But if $x_2 = 1$, then we should first perform an operation to cancel out the x_2 term in the second qubit.

The operation that will do it is

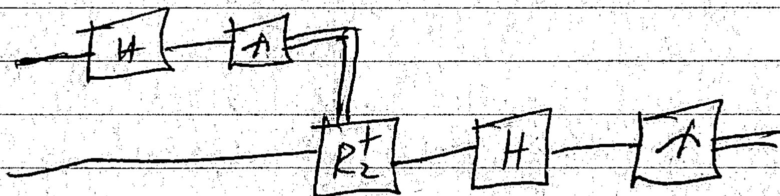
$$|0\rangle\langle 0| + e^{-2\pi i(0.01)}|1\rangle\langle 1|$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & e^{-2\pi i(0.01)} \end{bmatrix}$$

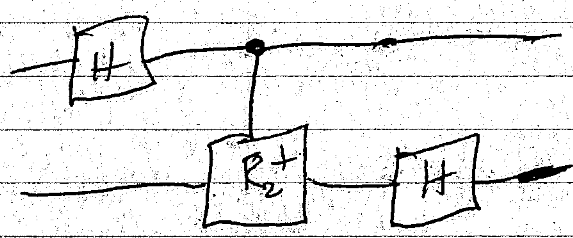
this is the inverse of a rotation operator that we will call R_2.

So the idea is that we could apply R_2^{-1} to the second qubit if $x_2 = 1$ + then apply a Hadamard to the second to get x_1 (measure in comp. basis)

So the circuit would look something like this



But we can implement this coherently as



(In general, we will need to maintain superpositions)

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consider how this generalizes to three qubits:

$$\frac{1}{\sqrt{2^3}} \sum_{y=0}^{2^3-1} e^{2\pi i (0.x_1x_2x_3)} |y\rangle =$$

$$\left(\frac{|0\rangle + e^{2\pi i (0.x_3)}|1\rangle}{\sqrt{2}} \right) \otimes \left(\frac{|0\rangle + e^{2\pi i (0.x_2x_3)}|1\rangle}{\sqrt{2}} \right) \otimes \left(\frac{|0\rangle + e^{2\pi i (0.x_1x_2x_3)}|1\rangle}{\sqrt{2}} \right)$$

Define $R_k = \begin{bmatrix} 1 & 0 \\ 0 & e^{\frac{2\pi i}{2^k}} \end{bmatrix}$

so that

$$R_k^{-1} : |0\rangle \to |0\rangle$$

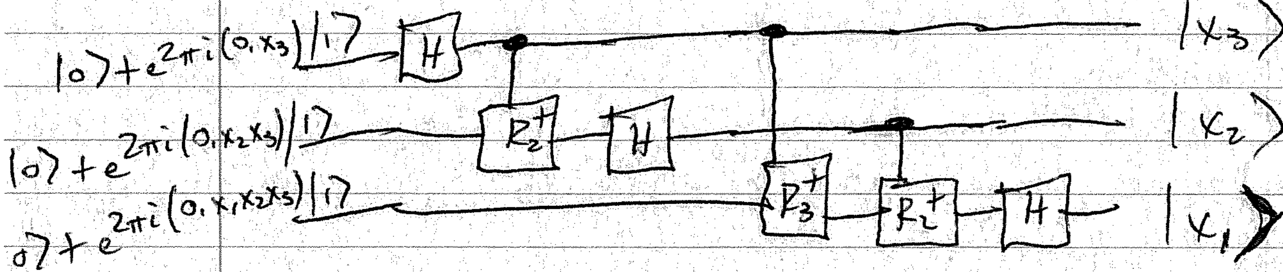
$$R_k^{-1} : |1\rangle \to e^{-2\pi i (0.00\ldots01)}|1\rangle$$

k^{th} position

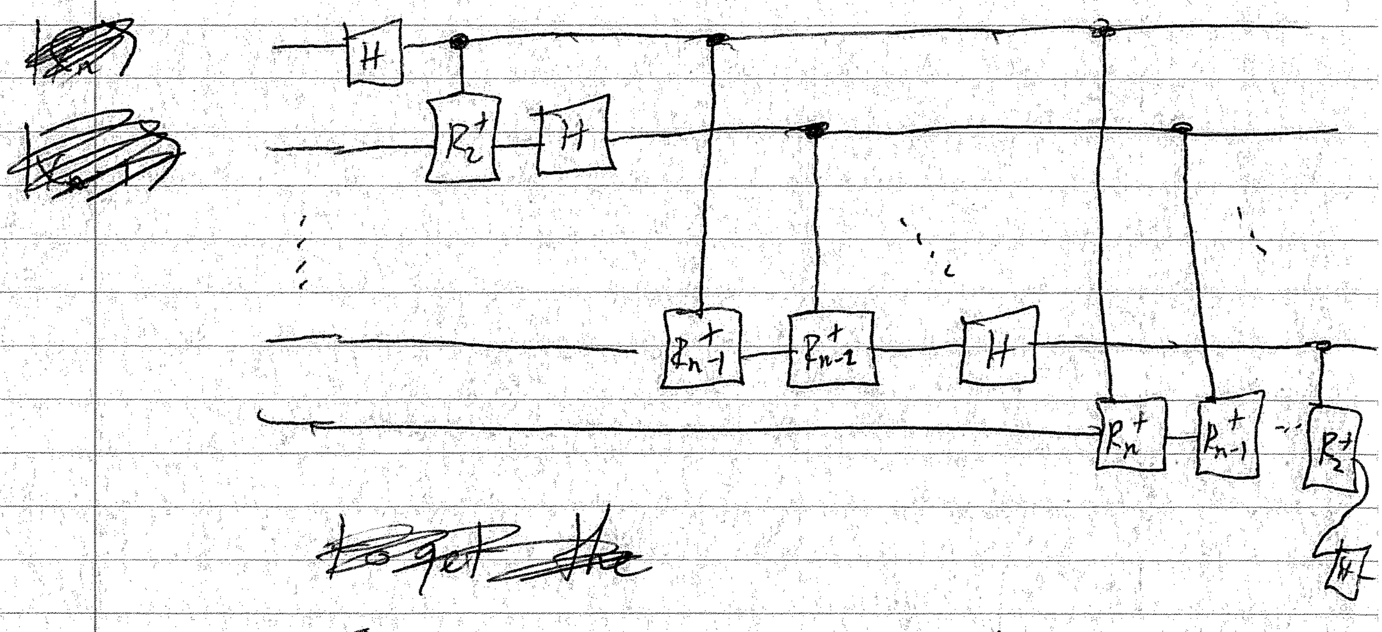
Then the idea is similar.

1) get x_3 by Hadamard + measurement
2) conditionally rotate the second qubit to eliminate x_3 + then get x_2
3) conditionally rotate the third qubit to eliminate both $x_2 + x_3$ + then get x_1

Done coherently, the circuit looks like this



Should be clear now how to generalize this. ~~Furthe~~



~~To get the~~

This circuit actually implements the inverse Fourier transform. To get the QFT, run this circuit backwards

~~The main~~ observe that the number of gates required is $O(n^2)$, which is quite favorable.

The main application of the QFT is in eigenvalue estimation

Suppose we are given a quantum circuit to implement a unitary U & an eigenstate $|\psi\rangle$ w/ eigenvalue $e^{2\pi i \omega}$

We would like to estimate ω.

We can then make controlled-U (c-U)

which acts as

$$
\begin{aligned}
c\text{-}U\,|1\rangle|\psi\rangle &= |1\rangle U|\psi\rangle \\
&= |1\rangle e^{2\pi i \omega}|\psi\rangle \\
&= e^{2\pi i \omega}|1\rangle|\psi\rangle
\end{aligned}
$$

Now if we prepare the control qubit in a superposition, then

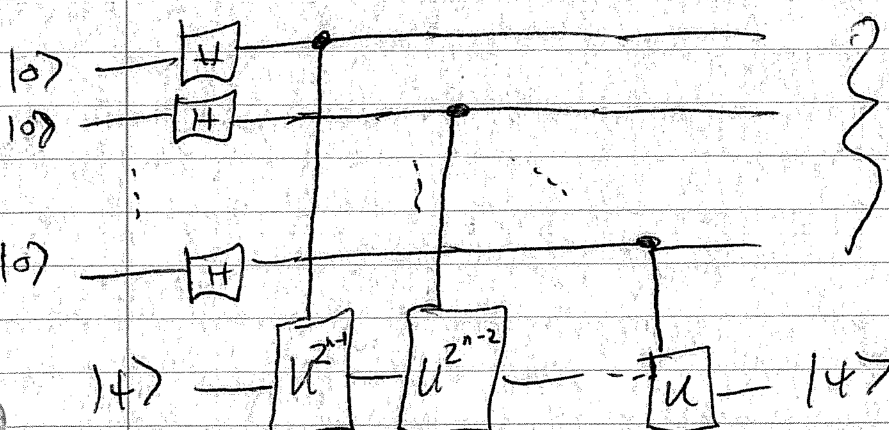
$$c\text{-}U \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)|\psi\rangle = \left(\frac{|0\rangle + e^{2\pi i \omega}|1\rangle}{\sqrt{2}}\right)|\psi\rangle$$

But we could also do

$$c\text{-}U^{2^j} \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)|\psi\rangle =$$

$$\left(\frac{|0\rangle + e^{2\pi i 2^j \omega}|1\rangle}{\sqrt{2}}\right)|\psi\rangle$$

In this way we can do



} Fourier transformed state

do QFT^{-1} to get estimate of ω