PHYS 7895 Spring 2014
Theory of Quantum Computation
Homework 1

Due Friday 21 February 2014, by 3pm in Nicholson 447
You are allowed to work with others as long as you write down who your collaborators are. The expectation is that this system will not be abused (i.e., you try all the problems first on your own and then discuss with collaborators after doing so.) Any detection of copying of solutions will be penalized with no credit for the assignment. Any late assignments will be penalized in the amount of 25% per day late.

1. Turing Machines:

   (a) Prove: For every function $f : \{0,1\}^* \to \{0,1\}$ and time-constructible function $T : \mathbb{N} \to \mathbb{N}$, if f is computable in time $T(n)$ by a Turing machine with tape alphabet Γ, then it is computable in time $4\log|\Gamma|T(n)$ by a Turing machine with tape alphabet $\{\triangleright, 0, 1, b\}$.

   (b) Prove: For every $f : \{0,1\}^* \to \{0,1\}$, time-constructible $T : \mathbb{N} \to \mathbb{N}$, if f is computable in time $T(n)$ by a Turing machine using k tapes (plus additional input and output tapes) then it is computable in time $5kT(n)^2$ by a Turing machine using only a single work tape (plus additional input and output tapes).

   (c) Explain why the complexity class P does not change under these low-level machine details.

2. A language $L \subseteq \{0,1\}^*$ is *Turing-recognizable* if for all $x \in L$, there exists a Turing machine that accepts on input x. A language $L \subseteq \{0,1\}^*$ is *Turing-decidable* if for all $x \in L$, there exists a Turing machine that accepts on input x and otherwise rejects. Let

   $$A_{TM} = \{(\langle M \rangle, x) : \text{ Turing machine } M \text{ accepts on input } x\}.$$

   (a) Prove or disprove: A_{TM} Turing-recogizable.

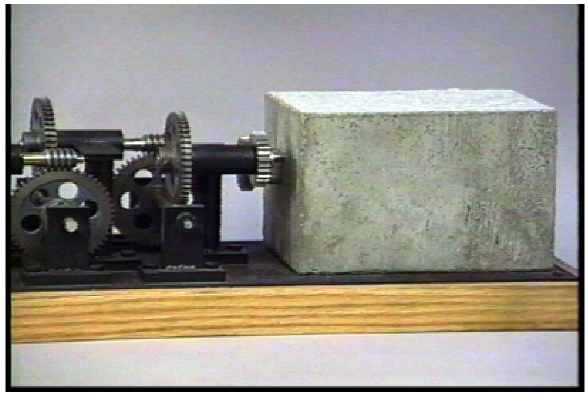
   (b) Prove or disprove: A_{TM} Turing-decidable.

3. Prove that P $\neq$ EXP by considering a "scaled down" version of the halting problem.

4. Prove that if NP $\neq$ coNP, then P $\neq$ NP.

5. The MIT museum contains a kinetic sculpture by Arthur Ganson called "Machine with concrete" (see figure below). It consists of 13 gears connected to one another in a series such that each gear moves 50 times slower than the previous one. The fastest gear is constantly rotated by an engine at a rate of 212 rotations per minute. The slowest gear is fixed to a block of concrete and so apparently cannot move at all. How come this machine does not break apart?

6. Max-k-2SAT and 2-Local Hamiltonian.



(a) Max-k-2SAT is defined to be the following decision problem: Given is a list of binary variables $x_1, \ldots, x_n$ and a Boolean formula consisting of m clauses of at most 2 literals each. The goal is to decide if there is an assignment of the variables such that at most k of the clauses are satisfied. For example, a particular *instance* of this decision problem could be x_1, x_2, x_3, x_4 along with the Boolean formula:

$$(x_1 \vee \overline{x_2}) \wedge (x_3 \vee x_4) \wedge (x_1 \vee \overline{x_3}) \wedge (\overline{x_2} \vee x_3) \wedge (x_3 \vee x_4)$$

with the goal of deciding whether three of the clauses can be satisfied. Prove that Max-k-2SAT is NP-complete.

(b) The quantum generalization of the above decision problem is called 2-Local Hamiltonian. Given is a list $\left\{H_i^{(k,l)}\right\}$ of n 4×4 Hermitian matrices each acting on at most two qubits (qubits k and l), such that the Hamiltonian for the n qubits is given as the sum of these matrices:

$$\sum_i H_i^{(k,l)},$$

where it is implicit that each matrix $H_i^{(k,l)}$ acts nontrivially on qubits k and l and trivially on the others (for example $H_i^{(1,2)} = (H_i)_{12} \otimes I_3 \otimes I_4 \otimes \cdots \otimes I_n$). Prove that 2-Local Hamiltonian is NP-hard.

7. A set of linear inequalities with rational coefficients over variables $u_1, \ldots, u_n$ is in IPROG (Integer Program) if there is an assignment of integer numbers in $\{0, 1, 2, \ldots\}$ to $u_1, \ldots, u_n$ that satisfies it. Prove that IPROG is NP-complete. (Hint: Consider a reduction from 3SAT.)

8. NP and coNP:

(a) Prove that the following language is coNP-complete:

TAUTOLOGY $= \{\varphi : \varphi$ is a Boolean formula that is satisfied by every assignment$\}$.

(b) Show that NP = coNP iff 3SAT and TAUTOLOGY are polynomial-time reducible to one another.

(c) Prove that P $\subseteq$ NP $\cap$ coNP.