**PHYS 7895: Gaussian Quantum Information**

# Lecture 16

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## 1 Overview

In the last lecture we discussed characteristic and Wigner functions of Gaussian states, and overlap formula for Gaussian states. We also discussed Gaussian quadratic evolutions.

In this lecture we will talk about Gaussian unitaries, and symplectic singular value decomposition of a symplectic matrix.

## 2 Gaussian unitaries

In the last lecture, we saw that any quadratic evolution takes a Gaussian state to another Gaussian state. As an example, let us see the action of displacement operators.

### 2.1 Displacement operator

Remember that a displacement operator is defined as

$$\hat{D}_r = e^{ir^T\Omega\hat{r}} \text{ for r} \in \mathbb{R}^{2\text{n}}. \tag{1}$$

Let us see the effect of this operator on the covariance matrix and mean vector of a Gaussian state $\rho$ with a covariance matrix $\sigma$ and mean vector $\bar{r}$. The new mean vector is:

$$\begin{aligned} \text{Tr}\left[\hat{r}\hat{D}_r^\dagger\rho\hat{D}_r\right] &= \text{Tr}\left[\hat{D}_r\hat{r}\hat{D}_r^\dagger\rho\right], & (2)\\ &= \text{Tr}\left[(\hat{r}+r)\rho\right], & (3)\\ &= \text{Tr}\left[\hat{r}\rho\right] + r, & (4)\\ &= \bar{r} + r. & (5) \end{aligned}$$

Since the covariance matrix is defined with the centered quadrature operators, the covariance matrix does not change under displacements. Since the first moments can be arbitrarily changed, no properties depending only on the spectrum of a gaussian state can depend on first moments. Examples of such properties include entropies and Renyi entropies. Furthermore, multimode displacements are tensor products of single mode displacements. Therefore, any quantity invariant under local unitaries does not depend on first moments. Examples of such quantities are mutual information and entanglement measures. Such linear displacements can be implemented by passive quadratic operations acting on the input modes and with external modes prepared in strong coherent states.

## 3 Symplectic operations

For a purely quadratic Hamiltonian, a Gaussian state evolves unitarily with the unitary operator

$$U = e^{\frac{i}{2}\hat{r}^T H \hat{r}}. \tag{6}$$

In this case, the covariance matrix and the mean vector of a generic state evolves as

$$\bar{r} \to S\bar{r}, \tag{7}$$

$$\sigma \to S\sigma S^T. \tag{8}$$

where $S = e^{\Omega H}$. These relations completely characterize the evolution of a Gaussian state.

## 4 Singular value decomposition of a symplectic transformation

**Theorem 1:** Any 2n × 2n symplectic matrix S can be decomposed as

$$S = O_1 Z O_2, \tag{9}$$

where $O_1$ and $O_2$ are symplectic and orthogonal, and

$$Z = \bigoplus_{j=1}^{n} \begin{bmatrix} z_j & 0 \\ 0 & z_j^{-1} \end{bmatrix},\ z_j \geq 1. \tag{10}$$

Here, the orthogonal symplectic matrices $O_1$ and $O_2$ represent passive optical elements such as beamsplitters and phase shifters, and $Z$ represents a single mode squeezer.

**Proof:** For this proof, use xp- ordering of the quadrature operators i.e. $[\hat{x}_1, \ldots \hat{x}_n, \hat{p}_1, \ldots \hat{p}_n]$. In this convention, the symplectic form has the structure:

$$\Omega \to J = \begin{bmatrix} 0_n & I_n \\ -I_n & 0_n \end{bmatrix} \tag{11}$$

First, we know that any symplectic matrix $S$ is invertible, and has the polar decomposition

$$S = PO \tag{12}$$

where

$$P = \left(SS^T\right)^{\frac{1}{2}} \tag{13}$$

$$O = \left(SS^T\right)^{-\frac{1}{2}} S \tag{14}$$

where $P$ is positive definite and $S$ is orthogonal. The orthogonality of $S$ can be seen as

$$OO^T = \left(SS^T\right)^{-\frac{1}{2}} \left(SS^T\right) \left(SS^T\right)^{-\frac{1}{2}} = I \tag{15}$$

**Lemma 1:** *The polar decomposition of a symplectic matrix $S$ as $S = PO$ is unique.*

To see that the polar decomposition of $S$ is unique, suppose that

$$S = P_1 O_1 = P_2 O_2. \tag{16}$$

We will first prove that $P_1^2 = P_2^2$ ; and since the square root of a positive definite matrix is unique, we will be able to show that $P_1 = P_2$. Let us first find an expression for $P_1^2$.

Rewriting (16), we have

$$P_1 = P_2 O_2 O_1^T \tag{17}$$

Right multiplying by $P_1$, we obtain

$$P_1^2 = P_2 O_2 O_1^T P_1 \tag{18}$$

Now we will find an expression for $P_2^2$. Rewriting (16) again, we have

$$P_2 = P_1 O_1 O_2^T \tag{19}$$

Right multiplying by $P_2$, we obtain

$$P_2^2 = P_1 O_1 O_2^T P_2 \tag{20}$$

Since $P_2$ is symmetric, we can write

$$P_2^2 = \left(P_2^2\right)^T \tag{21}$$
$$= P_2 O_2 O_1^T P_1 \tag{22}$$
$$= P_1^2 \tag{23}$$

wherein we used (18) for the last equality. Thus, $P_1 = P_2$. Since $P_1$ and $P_2$ are invertible, this also means that $O_1 = O_2$.

**Lemma 2:** *For a symplectic matrix $S$ with polar decomposition $S$=$PO$, $P$ & $O$ are symplectic.*

We will now show that not only $S$, but also $P$ and $O$ are symplectic. Consider that

$$S^T J S = J \implies S = J^{-1} S^{-T} J \tag{24}$$

Using the polar decomposition of $S$, we can rewrite the above as

$$S = J^{-1}(PO)^{-T} J \tag{25}$$
$$= J^{-1} P^{-T} O^{-T} J \tag{26}$$
$$= \underbrace{J^{-1} P^{-T} J}_{\text{positive definite}} \; \underbrace{J^{-1} O^{-T} J}_{\text{orthogonal}} \tag{27}$$

Because the polar decomposition is unique, we can write

$$P = J^{-1} P^{-T} J \implies P J P^T = J \tag{28}$$
$$O = J^{-1} O^{-T} J \implies O J O^T = J \tag{29}$$

That is, $P$ is symplectic, as well as $O$.

Finally, we can use these results to arrive at the conclusion for symplectic singular value decomposition.

Since $P$ is positive definite, it can be diagonalized with strictly positive eigenvalues. Suppose that $v$ is an eigenvector of $P$ with eigenvalue $\lambda$ i.e.

$$Pv = \lambda v \tag{30}$$

then $Jv$ is an eigenvector of $P$ with eigenvalue $\lambda^{-1}$ i.e.

$$PJv = \lambda^{-1} v \tag{31}$$

This is because

$$\begin{aligned} PJv &= PJP^T P^{-T} v && (32)\\ &= JP^{-T} v && (33)\\ &= JP^{-1} v && (34)\\ &= \lambda^{-1} Jv && (35) \end{aligned}$$

where we have used that $P$ is symplectic and symmetric. So this means that the eigenvalues of $P$ come in pairs, and one is the inverse of the other. Also,

$$v^T Jv = 0 \; \forall \, v. \tag{36}$$

So,

$$v \perp Jv \; \forall \, v. \tag{37}$$

Define V as the 2n $\times$ n matrix of the first n eigenvectors of P. Then JV is 2n $\times$ n matrix of the next n eigenvectors of P. Using these matrices, we can write the eigendecomposition of P as

$$P = \underbrace{\begin{bmatrix} JV & V \end{bmatrix}}_{K} \begin{bmatrix} \lambda_1^{-1} & & & & & \\ & \ddots & & & & \\ & & \lambda_n^{-1} & & & \\ & & & \lambda_1 & & \\ & & & & \ddots & \\ & & & & & \lambda_n \end{bmatrix} \begin{bmatrix} V^T J^T \\ V^T \end{bmatrix} \equiv K \begin{bmatrix} \Lambda^{-1} & 0 \\ 0 & \Lambda \end{bmatrix} K^T \tag{38}$$

where we defined $\Lambda = diag(\lambda_1 \dots \lambda_n)$ with $\lambda_i \geq 1$. K itsef is symplectic and orthogonal. To see this, notice that

$$\begin{aligned} K^T JK &= \begin{bmatrix} V^T J^T \\ V^T \end{bmatrix} J \begin{bmatrix} JV & V \end{bmatrix} && (39)\\ &= \begin{bmatrix} V^T J^T J = V^T \\ V^T J \end{bmatrix} \begin{bmatrix} JV & V \end{bmatrix} && (40)\\ &= \begin{bmatrix} V^T JV & V^T V \\ V^T JJV & V^T JV \end{bmatrix} && (41)\\ &= \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} && (42)\\ &= J && (43) \end{aligned}$$

where the diagonal entries of the matrix in Eq. (42) is zero because of Eq. (37). Finally we can write

$$
\begin{aligned}
S &= PO \qquad (44)\\
&= K \begin{bmatrix} \Lambda^{-1} & 0 \\ 0 & \Lambda \end{bmatrix} K^T O \qquad (45)
\end{aligned}
$$

with $P$ positive definite and symplectic, and $O$ orthogonal and symplectic. Now, identify

$$
\begin{aligned}
O_1 &\equiv K, \qquad (46)\\
Z &\equiv \begin{bmatrix} \Lambda^{-1} & 0 \\ 0 & \Lambda \end{bmatrix}, \qquad (47)\\
O_2 &\equiv K^T O. \qquad (48)
\end{aligned}
$$

Thus we have

$$S = O_1 Z O_2 \qquad (49)$$