PHYS 7895: Gaussian Quantum Information

Lecture 13

Lecturer: Mark M. Wilde *Scribe: Kevin Valson Jacob*

1 Overview

In the last lecture we derived formulas for various overlap measures of Gaussian states.

In this lecture we discuss quasiprobability distributions and characteristic functions to describe Gaussian states. We will also discuss various properties of displacement operators and how they can be used to describe any state.

2 Characteristic functions and Quasiprobability distributions

While Gaussian states can be described by their covariance matrix and mean vector, an alternative way to visualize them is in the phase space. Wigner functions are well-known quasiprobability distributions that can fully characterize a state. For Gaussian states, the Wigner function is non-negative.

Any quantum mechanical process has three parts: state preparation, subsequent evolution through a channel, and finally measurements. This can be captured via a quasi-probability distribution as

$$\mathrm{Tr}[\Omega\mathcal{N}(\rho)] = \int \underbrace{W(\Omega|\lambda)}_{\text{Measurement}} \underbrace{W_{\mathcal{N}}(\lambda|\lambda')}_{\text{Channel}} \underbrace{W_{\rho}(\lambda')}_{\text{State}} d\lambda\, d\lambda' \tag{1}$$

If each of the three terms of the integrand are positive, then it means that there is an underlying classical description of the quantum physical experiment.

3 Displacement operators

Recall that we defined the displacement operator as

$$\hat{D}_{-r} = e^{-ir^T\Omega\hat{r}} \tag{2}$$

where

$$r = \begin{pmatrix} x \\ p \end{pmatrix} \tag{3}$$

Alternatively, we can define the displacement operator using complex numbers and mode creation and annihilation operators as

$$\hat{D}_{\alpha} = e^{\alpha\hat{a}^{\dagger} - \alpha^{*}\hat{a}} \tag{4}$$

where

$$\alpha = \frac{x + ip}{\sqrt{2}} \tag{5}$$

Then

$$\hat{D}_{-r} = \hat{D}_{\alpha} \tag{6}$$

Coherent states are displaced states of the vacuum as

$$\hat{D}_{\alpha}|0\rangle = |\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \tag{7}$$

Successive displacements are equivalent to a single displacement up to an overall phase factor as

$$\hat{D}_{\alpha}\hat{D}_{\beta} = e^{\frac{1}{2}(\alpha\beta^* - \alpha^*\beta)} \hat{D}_{\alpha+\beta} \tag{8}$$

This allows us to compute the overlap of two coherent states. The overlap of two coherent states is always strictly positive and is given as

$$\begin{aligned}
\langle\beta|\alpha\rangle &= \langle 0|\hat{D}_{-\beta}\hat{D}_{\alpha}|0\rangle && (9)\\
&= \langle 0|\hat{D}_{\alpha-\beta}|0\rangle e^{\frac{1}{2}(\alpha\beta^* - \alpha^*\beta)} && (10)\\
&= \langle 0|\alpha - \beta\rangle e^{\frac{1}{2}(\alpha\beta^* - \alpha^*\beta)} && (11)\\
&= e^{-\frac{1}{2}|\alpha-\beta|^2} e^{\frac{1}{2}(\alpha\beta^* - \alpha^*\beta)} && (12)
\end{aligned}$$

Coherent states together form an overcomplete basis as

$$\hat{I} = \frac{1}{\pi} \int d^2\alpha \ |\alpha\rangle\langle\alpha| \tag{13}$$

This fact can be used to evaluate the traces of trace-class operators. For any trace-class operator, we have

$$\begin{aligned}
\mathrm{Tr}[\hat{O}] &= \sum_{m=0}^{\infty} \langle m|\hat{O}|m\rangle && (14)\\
&= \sum_{m=0}^{\infty} \langle m|\frac{1}{\pi} \int d^2\alpha \ |\alpha\rangle\langle\alpha|\hat{O}|m\rangle && (15)\\
&= \frac{1}{\pi} \int d^2\alpha \sum_{m=0}^{\infty} \langle m|\alpha\rangle\langle\alpha|\hat{O}|m\rangle && (16)\\
&= \frac{1}{\pi} \int d^2\alpha \sum_{m=0}^{\infty} \langle\alpha|\hat{O}|m\rangle\langle m|\alpha\rangle && (17)\\
&= \frac{1}{\pi} \int_{\mathbb{C}} d^2\alpha \ \langle\alpha|\hat{O}|\alpha\rangle && (18)
\end{aligned}$$

3.1 Mean vector of coherent states

The above result can be used to evaluate the mean vector and covariance matrix of a coherent state. Alternatively, there is a simpler derivation:

$$
\begin{aligned}
\mathrm{Tr}\Big[\hat{x}|\alpha\rangle\langle\alpha|\Big] &= \langle\alpha|\hat{x}|\alpha\rangle && (19)\\
&= \langle\alpha|\frac{\hat{a}+\hat{a}^\dagger}{\sqrt{2}}|\alpha\rangle && (20)\\
&= \frac{\langle\alpha|\left(\hat{a}|\alpha\rangle\right)+\left(\langle\alpha|\hat{a}^\dagger\right)|\alpha\rangle}{\sqrt{2}} && (21)\\
&= \frac{\langle\alpha|\alpha|\alpha\rangle+\langle\alpha|\alpha^*|\alpha\rangle}{\sqrt{2}} && (22)\\
&= \frac{2\mathrm{Re}\{\alpha\}}{\sqrt{2}} && (23)
\end{aligned}
$$

Alternatively, we can use

$$
\begin{aligned}
\mathrm{Tr}\Big[\hat{x}|\alpha\rangle\langle\alpha|\Big] &= \mathrm{Tr}\left[\hat{x}\hat{D}_\alpha|0\rangle\langle 0|\hat{D}_{-\alpha}\right] && (24)\\
&= \mathrm{Tr}\left[\hat{D}_{-\alpha}\hat{x}\hat{D}_\alpha|0\rangle\langle 0|\right] && (25)\\
&= \mathrm{Tr}\left[\hat{D}_r\hat{x}\hat{D}_{-r}|0\rangle\langle 0|\right] && (26)\\
&= \mathrm{Tr}\Big[(\hat{x}+x)|0\rangle\langle 0|\Big] && (27)\\
&= \langle 0|\hat{x}|0\rangle + x && (28)\\
&= x && (29)\\
&= \sqrt{2}\mathrm{Re}(\alpha) && (30)
\end{aligned}
$$

Similarly, we can find the expectation value of momentum quadrature also. Thus we obtain that for a coherent state $|\alpha\rangle$, the mean vector $\overline{r}$ is

$$
\overline{r} = \begin{pmatrix}\sqrt{2}\mathrm{Re}(\alpha)\\ \sqrt{2}\mathrm{Im}(\alpha)\end{pmatrix}, \tag{31}
$$

3.2 Covariance matrix of coherent states

We now calculate the covariance matrix of coherent states. We note that

$$
\begin{aligned}
\mathrm{Tr}\left[\hat{x}^2|\alpha\rangle\langle\alpha|\right] &= \mathrm{Tr}\left[\hat{x}^2\hat{D}_\alpha|0\rangle\langle 0|\hat{D}_{-\alpha}\right] && (32)\\
&= \mathrm{Tr}\left[\hat{D}_r\hat{x}^2\hat{D}_{-r}|0\rangle\langle 0|\right] && (33)\\
&= \mathrm{Tr}\left[(\hat{x}+x)^2|0\rangle\langle 0|\right] && (34)\\
&= \mathrm{Tr}\left[(\hat{x}^2+2x\hat{x}+x^2)|0\rangle\langle 0|\right] && (35)\\
&= 1/2 + x^2 && (36)
\end{aligned}
$$

Therefore we have

$$2\ \mathrm{Tr}\left[(\hat{x}-x)^2|\alpha\rangle\langle\alpha|\right] = 1 \tag{37}$$

Similarly, we can find the other covariance matrix elements also to find that the covariance matrix σ is

$$\sigma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{38}$$

We note that the covariance matrix of coherent states is the same as that of the vacuum state.

3.3 Trace of a displacement operator

The displacement operator is not trace class, but it is useful to consider its trace in a generalized sense as follows:

$$\mathrm{Tr}[\hat{D}(\beta)] = \pi\delta^2(\beta), \beta \in \mathbb{C}, \tag{39}$$

which turns out to be a key tool in continuous-variable quantum information.

To prove this, we will first find out $\langle\alpha|\hat{D}(\beta)|\alpha\rangle$. This is

$$\begin{aligned}
\langle\alpha|\hat{D}(\beta)|\alpha\rangle &= \langle 0|\hat{D}^\dagger(\alpha)\hat{D}(\beta)\hat{D}(\alpha)|0\rangle & (40)\\
&= \langle 0|\hat{D}^\dagger(\alpha)\mathrm{exp}\left(\beta\hat{a}^\dagger - \beta^*\hat{a}\right)\hat{D}(\alpha)|0\rangle & (41)\\
&= \langle 0|\mathrm{exp}\left[\hat{D}^\dagger(\alpha)(\beta\hat{a}^\dagger - \beta^*\hat{a})\hat{D}(\alpha)\right]|0\rangle & (42)\\
&= \langle 0|\mathrm{exp}\left[\beta(\hat{a}^\dagger + \alpha) - \beta^*(\hat{a}+\alpha)\right]|0\rangle & (43)
\end{aligned}$$

Now we use the results that

$$\begin{aligned}
D^\dagger(\alpha)\hat{a}^\dagger D(\alpha) &= \hat{a}^\dagger + \alpha^*, & (44)\\
D^\dagger(\alpha)\hat{a}D(\alpha) &= \hat{a} + \alpha & (45)
\end{aligned}$$

so as to obtain

$$\begin{aligned}
\langle\alpha|\hat{D}(\beta)|\alpha\rangle &= e^{\beta\alpha^* - \beta^*\alpha}\langle 0|\hat{D}(\beta)|0\rangle & (46)\\
&= e^{\beta\alpha^* - \beta^*\alpha}\langle 0|\beta\rangle & (47)\\
&= e^{\beta\alpha^* - \beta^*\alpha}e^{-\frac{1}{2}|\beta|^2} & (48)
\end{aligned}$$

Using the above result, we can find an expression for the trace of a displacement operator as

$$\begin{aligned}
\mathrm{Tr}[\hat{D}(\beta)] &= \frac{1}{\pi}\int d^2\alpha\ \langle\alpha|\hat{D}(\beta)|\alpha\rangle & (49)\\
&= \frac{e^{-\frac{|\beta|^2}{2}}}{\pi}\int d^2\alpha\ e^{\beta\alpha^* - \beta^*\alpha} & (50)
\end{aligned}$$

To simplify, redefine

$$\alpha = x + iy, \qquad \beta = u + iv, \tag{51}$$

so that $\beta\alpha^* - \beta^*\alpha = 2i(vx - uy)$. Now we have

$$\begin{aligned}
\mathrm{Tr}[\hat{D}(\beta)] &= \frac{e^{-\frac{|\beta|^2}{2}}}{\pi} \int\int dxdy\ e^{2ivx-2iuy} && (52)\\
&= \frac{e^{-\frac{|\beta|^2}{2}}}{\pi} \int dx\ e^{2ivx} \int dy\ e^{-2iuy} && (53)\\
&= \frac{e^{-\frac{|\beta|^2}{2}}}{\pi} 2\pi\,\delta(2v)\,2\pi\,\delta(2u) && (54)\\
&= \frac{e^{-\frac{|\beta|^2}{2}}}{\pi} \pi^2\delta(v)\delta(u) && (55)\\
&= \pi\delta^2(\beta) && (56)
\end{aligned}$$

as the exponential factor $e^{-\frac{|\beta|^2}{2}}$ is equal to one when the delta function is nonzero at $\beta = 0$.

3.4 Hilbert-Schmidt inner product of displacement operators

Now we will find the Hilbert-Schmidt inner product of displacement operators. Using the above relation, this is

$$\begin{aligned}
\mathrm{Tr}[\hat{D}(\alpha)\hat{D}(-\beta)] &= e^{\frac{1}{2}(-\alpha\beta^*+\alpha^*\beta)}\mathrm{Tr}[\hat{D}(\alpha-\beta)] && (57)\\
&= e^{\frac{1}{2}(-\alpha\beta^*+\alpha^*\beta)}\pi\delta^2(\alpha-\beta) && (58)\\
&= \pi\delta^2(\alpha-\beta) && (59)
\end{aligned}$$

wherein again, we use the fact that the exponential factor $e^{\frac{1}{2}(-\alpha\beta^*+\alpha^*\beta)}$ is equal to one when the delta function is nonzero at $\alpha = \beta$.

In terms of real variables, we can write this orthogonality relation as

$$\mathrm{Tr}\left[\hat{D}_r\hat{D}_{-s}\right] = 2\pi\delta^2\left(r-s\right) \tag{60}$$

where

$$\alpha = \frac{x_r + ip_r}{\sqrt{2}}, \qquad \beta = \frac{x_s + ip_s}{\sqrt{2}}. \tag{61}$$

Generalizing to n modes, the orthogonality relations are as follows:

$$\begin{aligned}
\mathrm{Tr}\left[\hat{D}\left(\underline{\alpha}\right)\hat{D}\left(-\underline{\beta}\right)\right] &= \pi^n\delta^{2n}(\underline{\alpha}-\underline{\beta}), && (62)\\
\mathrm{Tr}\left[\hat{D}_{\underline{r}}\hat{D}_{-\underline{s}}\right] &= (2\pi)^n\delta^{2n}(\underline{r}-\underline{s}). && (63)
\end{aligned}$$

4 Characteristic functions

We define the symmetrically ordered Weyl characteristic function of a state ρ as

$$\chi_\rho(\alpha) = \mathrm{Tr}[\hat{D}(\alpha)\rho] \qquad \forall\alpha\in\mathbb{C} \tag{64}$$

This characteristic function is finite for all $\alpha \in \mathbb{C}$ if ρ is trace class. To see this we use the Hölder inequality to have

$$\left|\mathrm{Tr}\left[\hat{D}(\alpha)\rho\right]\right| \leq \left\|\hat{D}(\alpha)\right\|_\infty \|\rho\|_1 = \|\rho\|_1 < \infty. \tag{65}$$

Using the Weyl characteristic function, we can write the state as

$$\rho = \frac{1}{\pi}\int d^2\alpha\ \chi_\rho(\alpha)\hat{D}(-\alpha). \tag{66}$$

To prove this, we note that

$$\begin{align}
\rho &= \left[\frac{1}{\pi}\int d^2\alpha\ |\alpha\rangle\langle\alpha|\right]\rho\left[\frac{1}{\pi}\int d^2\beta\ |\beta\rangle\langle\beta|\right] \tag{67}\\
&= \frac{1}{\pi^2}\int\int d^2\alpha\ d^2\beta\ \langle\alpha|\rho|\beta\rangle\ |\alpha\rangle\langle\beta|. \tag{68}
\end{align}$$

So we only have to show that

$$|\alpha\rangle\langle\beta| = \frac{1}{\pi}\int d^2\gamma\ \mathrm{Tr}\left[|\alpha\rangle\langle\beta|\hat{D}_\gamma\right]\hat{D}_{-\gamma} \tag{69}$$

This can be rewritten as

$$\begin{align}
|0\rangle\langle 0| &= \frac{1}{\pi}\int d^2\gamma\ \mathrm{Tr}\left[|\alpha\rangle\langle\beta|\hat{D}_\gamma\right]\hat{D}_{-\alpha}\hat{D}_{-\gamma}\hat{D}_\beta \tag{70}\\
&= \frac{1}{\pi}\int d^2\gamma\ \langle\beta-\gamma|\alpha\rangle e^{\gamma\beta^*-\gamma^*\beta}\hat{D}_{-\alpha}\hat{D}_{-\gamma}\hat{D}_\beta \tag{71}\\
&= \frac{1}{\pi}\int d^2\gamma\ e^{-\frac{1}{2}|\beta-\alpha-\gamma|^2}\hat{D}_{\beta-\alpha-\gamma} \tag{72}\\
&= \frac{1}{\pi}\int d^2\gamma\ e^{-\frac{1}{2}|\gamma|^2}\hat{D}_\gamma \tag{73}
\end{align}$$

We will prove this in the next lecture.