PHYS 7895: Gaussian Quantum Information

Lecture 12

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1 Overview

In the last lecture we derived the formulas for the Rényi entropies, purity, and the entropy of Gaussian states.

In this lecture we derive the formulas for various overlap measures of two Gaussian states such as Holevo fidelity, Uhlmann fidelity, Petz-Rényi relative entropy, and sandwiched Rényi relative entropy.

2 Overlap formulas for Gaussian states

In quantum information, we are often interested in finding out how close two states are. A simple overlap formula between two states ρ and τ is $\mathrm{Tr}[\rho\tau]$. More generally, we compute overlap formulas of the following kind:

$$F_H(\rho,\tau) = \mathrm{Tr}\left[\sqrt{\rho}\sqrt{\tau}\right]^2 \tag{1}$$

$$F(\rho,\tau) = \left\|\sqrt{\rho}\sqrt{\tau}\right\|_1^2 = \mathrm{Tr}\left[\sqrt{\sqrt{\tau}\rho\sqrt{\tau}}\right]^2 \tag{2}$$

F_H represents the Holevo fidelity whereas F represents the Uhlmann fidelity. Generalizing the above, we are interested in Rényi overlaps of the following kind:

$$Q_\alpha(\rho,\tau) = \mathrm{Tr}\left[\rho^\alpha \tau^{1-\alpha}\right], \tag{3}$$

$$\widetilde{Q}_\alpha(\rho,\tau) = \mathrm{Tr}\left[(\tau^{\frac{1-\alpha}{2\alpha}}\rho\tau^{\frac{1-\alpha}{2\alpha}})^\alpha\right] \tag{4}$$

$$= \mathrm{Tr}\left[\left(\rho^{\frac{1}{2}}\tau^{\frac{1-\alpha}{\alpha}}\rho^{\frac{1}{2}}\right)^\alpha\right], \tag{5}$$

where $\alpha \in (0,1)\cup(1,\infty)$. Here, Q represents the Petz-Rényi relative entropy while $\widetilde{Q}$ represents the sandwiched Rényi relative entropy. Note that $Q_{\alpha=\frac{1}{2}}(\rho,\tau) = \sqrt{F_H(\rho,\tau)}$ and $\widetilde{Q}_{\alpha=\frac{1}{2}}(\rho,\tau) = \sqrt{F(\rho,\tau)}$. The reason these overlap functions are interesting is because we can bound the operationally meaningful trace distance between two states as

$$F_H(\rho,\tau) \le F(\rho,\tau), \tag{6}$$

$$1-\sqrt{F(\rho,\tau)} \le 1-\sqrt{F_H(\rho,\tau)} \le \frac{1}{2}\left\|\rho-\tau\right\|_1 \le \sqrt{1-F(\rho,\tau)} \le \sqrt{1-F_H(\rho,\tau)}. \tag{7}$$

As we do not possess a general formula for the trace distance between Gaussian states, the above relation proves useful in bounding it. For simplicity in calculating these expressions, we will restrict ourselves to consider only zero-mean states.

2.1 Simple overlap of Gaussian states

Let us first consider the simple overlap formula $\mathrm{Tr}[\rho\tau]$ for states

$$\rho = \frac{1}{\sqrt{\mathrm{Det}\left(\frac{\sigma_\rho + i\Omega}{2}\right)}} \exp\left(-\frac{1}{2}\hat{r}^T H_\rho \hat{r}\right), \tag{8}$$

$$\tau = \frac{1}{\sqrt{\mathrm{Det}\left(\frac{\sigma_\tau + i\Omega}{2}\right)}} \exp\left(-\frac{1}{2}\hat{r}^T H_\tau \hat{r}\right). \tag{9}$$

Thus we obtain

$$\mathrm{Tr}[\rho\tau] = \frac{1}{\sqrt{\mathrm{Det}\left(\frac{\sigma_\rho + i\Omega}{2}\right)\mathrm{Det}\left(\frac{\sigma_\tau + i\Omega}{2}\right)}} \mathrm{Tr}\left[e^{-\frac{1}{2}\hat{r}^T H_\rho \hat{r}} e^{-\frac{1}{2}\hat{r}^T H_\tau \hat{r}}\right] \tag{10}$$

We now wish to simplify the RHS of the above expression involving the product of two quadratic exponentials. In order to do so, we note the general result that for complex symmetric matrices H_1 and H_2, there exists another complex symmetric matrix H_3 such that if H_3 satisfies the relation

$$e^{-\frac{1}{2}\hat{r}^T H_1 \hat{r}} e^{-\frac{1}{2}\hat{r}^T H_2 \hat{r}} = e^{-\frac{1}{2}\hat{r}^T H_3 \hat{r}}, \tag{11}$$

then it also satisfies

$$e^{-i\Omega H_1} e^{-i\Omega H_2} = e^{-i\Omega H_3}. \tag{12}$$

The latter relation is useful in finding a form for H_3. Inverting the expression, we obtain

$$e^{i\Omega H_3} = e^{i\Omega H_2} e^{i\Omega H_1} \tag{13}$$

For simplicity in notations, define

$$W_3 = \left(I + e^{i\Omega H_3}\right)\left(I - e^{i\Omega H_3}\right)^{-1}, \tag{14}$$

$$\sigma_3 = -W_3 i\Omega. \tag{15}$$

The latter implies that

$$\sigma_3 = \coth\left(\frac{i\Omega H_3}{2}\right) i\Omega. \tag{16}$$

Also, we note that

$$\sigma_1 = \coth\left(\frac{i\Omega H_1}{2}\right) i\Omega, \tag{17}$$

$$\sigma_2 = \coth\left(\frac{i\Omega H_2}{2}\right) i\Omega. \tag{18}$$

Using these, we can arrive at the final form of σ_3 and H_3 as (see page 13, Ref. [1])

$$\sigma_3 = -i\Omega + (\sigma_2 + i\Omega)(\sigma_1 + \sigma_2)^{-1}(\sigma_1 + i\Omega), \tag{19}$$

$$H_3 = 2i\Omega \mathrm{arccoth}\left(\sigma_3 i\Omega\right), \tag{20}$$

from which we can obtain

$$\sqrt{\mathrm{Det}\left(\frac{\sigma_3 + i\Omega}{2}\right)} = \sqrt{\mathrm{Det}\left(\left(\frac{\sigma_2 + i\Omega}{2}\right)\left(\frac{\sigma_1 + \sigma_2}{2}\right)^{-1}\left(\frac{\sigma_1 + i\Omega}{2}\right)\right)} \tag{21}$$

Note that σ_3 is complex symmetric. It can be shown that (see Prop. 11, Ref. [1])

$$\mathrm{Tr}\left[e^{-\frac{1}{2}\hat{r}^T H_3 \hat{r}}\right] = \sqrt{\mathrm{Det}\left(\frac{\sigma_3 + i\Omega}{2}\right)} \tag{22}$$

Thus finally we can simplify the expression for the overlap as

$$\begin{aligned}
\mathrm{Tr}\left[\rho\tau\right] &= \frac{1}{\sqrt{\mathrm{Det}\left(\frac{\sigma_\rho + i\Omega}{2}\right)\mathrm{Det}\left(\frac{\sigma_\tau + i\Omega}{2}\right)}} \frac{\sqrt{\mathrm{Det}\left(\frac{\sigma_\rho + i\Omega}{2}\right)\mathrm{Det}\left(\frac{\sigma_\tau + i\Omega}{2}\right)}}{\sqrt{\mathrm{Det}\left(\frac{\sigma_\rho + \sigma_\tau}{2}\right)}} &(23)\\
&= \frac{1}{\sqrt{\mathrm{Det}\left(\frac{\sigma_\rho + \sigma_\tau}{2}\right)}} &(24)\\
&= \frac{2^n}{\sqrt{\mathrm{Det}(\sigma_\rho + \sigma_\tau)}} &(25)
\end{aligned}$$

We note that the overlap expression is not a function of the Hamiltonian matrix which implies that it is valid for pure (coherent) states also.

If the mean vectors of the states are represented by $\underline{r}_\rho$ and $\underline{r}_\tau$, then it can be shown that the overlap expression is

$$\mathrm{Tr}[\rho\tau] = \frac{2^n}{\sqrt{\mathrm{Det}(\sigma_\rho + \sigma_\tau)}} \exp[-\underline{\delta}^T(\sigma_\rho + \sigma_\tau)^{-1}\underline{\delta}] \tag{26}$$

where $\underline{\delta} = \underline{r}_\rho - \underline{r}_\tau$.

2.2 Petz-Rényi relative entropy of Gaussian states

Having computed an expression for the simple overlap formula, we now move on to compute the Petz-Rényi overlap of Gaussian states, defined as

$$Q_\alpha(\rho, \tau) = \mathrm{Tr}\left[\rho^\alpha \tau^{1-\alpha}\right]. \tag{27}$$

We will restrict ourselves to first consider the case when $\alpha \in (0, 1)$. For notational simplicity, label the normalization of Gaussian states as

$$Z_\rho \equiv \sqrt{\mathrm{Det}\left(\frac{\sigma_\rho + i\Omega}{2}\right)}, \tag{28}$$

For states ρ and τ as defined in (8) and (9), the Petz-Rényi overlap is

$$\begin{aligned}
Q_\alpha(\rho, \tau) &= \frac{1}{(Z_\rho)^\alpha (Z_\tau)^{1-\alpha}} \mathrm{Tr}\left[e^{-\frac{1}{2}\hat{r}^T \alpha H_\rho \hat{r}} e^{-\frac{1}{2}\hat{r}^T (1-\alpha) H_\tau \hat{r}}\right] &(29)\\
&= \frac{Z_{\rho(\alpha)} Z_{\tau(1-\alpha)}}{(Z_\rho)^\alpha (Z_\tau)^{1-\alpha}} \mathrm{Tr}\left[\frac{e^{-\frac{1}{2}\hat{r}^T \alpha H_\rho \hat{r}}}{Z_{\rho(\alpha)}} \frac{e^{-\frac{1}{2}\hat{r}^T (1-\alpha) H_\tau \hat{r}}}{Z_{\tau(1-\alpha)}}\right] &(30)
\end{aligned}$$

wherein we have used the fact that the Hamiltonian matrix of the exponent of a Gaussian state is the product of that exponent with the original Hamiltonian matrix. Now we can apply the simple overlap that we calculated earlier to obtain

$$Q_\alpha(\rho,\tau) = \frac{Z_{\rho(\alpha)} Z_{\tau(1-\alpha)}}{(Z_\rho)^\alpha (Z_\tau)^{1-\alpha}} \frac{1}{\sqrt{\mathrm{Det}\left(\frac{\sigma_{\rho(\alpha)}+\sigma_{\tau(1-\alpha)}}{2}\right)}}, \tag{31}$$

where

$$Z_{\rho(\alpha)} = \sqrt{\mathrm{Det}\left(\frac{\sigma_{\rho(\alpha)} + i\Omega}{2}\right)}, \tag{32}$$

$$Z_{\tau(1-\alpha)} = \sqrt{\mathrm{Det}\left(\frac{\sigma_{\tau(1-\alpha)} + i\Omega}{2}\right)}, \tag{33}$$

and

$$\sigma_{\rho(\alpha)} = \frac{\left[I + (\sigma_\rho i\Omega)^{-1}\right]^\alpha + \left[I - (\sigma_\rho i\Omega)^{-1}\right]^\alpha}{\left[I + (\sigma_\rho i\Omega)^{-1}\right]^\alpha - \left[I - (\sigma_\rho i\Omega)^{-1}\right]^\alpha} i\Omega, \tag{34}$$

$$\sigma_{\tau(1-\alpha)} = \frac{\left[I + (\sigma_\tau i\Omega)^{-1}\right]^{1-\alpha} + \left[I - (\sigma_\tau i\Omega)^{-1}\right]^{1-\alpha}}{\left[I + (\sigma_\tau i\Omega)^{-1}\right]^{1-\alpha} - \left[I - (\sigma_\tau i\Omega)^{-1}\right]^{1-\alpha}} i\Omega. \tag{35}$$

Holevo Fidelity: For $\alpha = \frac{1}{2}$ the above expression simplifies to

$$Q_{\frac{1}{2}}(\rho,\tau) = \sqrt{F_H(\rho,\tau)} = \mathrm{Tr}\left[\sqrt{\rho}\sqrt{\tau}\right] \tag{36}$$

$$= \frac{Z_{\rho(1/2)} Z_{\tau(1/2)}}{(Z_\rho)^{\frac{1}{2}} (Z_\tau)^{\frac{1}{2}}} \frac{1}{\sqrt{\mathrm{Det}\left(\frac{\sigma_{\rho(1/2)}+\sigma_{\tau(1/2)}}{2}\right)}} \tag{37}$$

where

$$\sigma_{\rho(1/2)} = \left(\sqrt{I + (\sigma_\rho \Omega)^{-2}} + I\right)\sigma_\rho, \tag{38}$$

$$\sigma_{\tau(1/2)} = \left(\sqrt{I + (\sigma_\tau \Omega)^{-2}} + I\right)\sigma_\tau, \tag{39}$$

and

$$Z_{\rho(1/2)} = \sqrt{\mathrm{Det}\left(\frac{\sigma_{\rho(1/2)} + i\Omega}{2}\right)}, \tag{40}$$

$$Z_{\tau(1/2)} = \sqrt{\mathrm{Det}\left(\frac{\sigma_{\tau(1/2)} + i\Omega}{2}\right)}. \tag{41}$$

Now we shall consider the case of $\alpha > 1$. This is interesting because we will have to deal with inverses of Gaussian states which are in general unbounded operators. However, we can find expressions for overlaps. In order to derive such an expression, note that for $H_1, H_2 > 0$ such that $\sigma_2 > \sigma_1$, we have (see Ref. [1])

$$\mathrm{Tr}\left[e^{-\hat{r}^T H_1 \hat{r}} e^{-\hat{r}^T(-H_2)\hat{r}}\right] = \frac{\sqrt{\mathrm{Det}\left(\frac{\sigma_1+i\Omega}{2}\right)\mathrm{Det}\left(\frac{\sigma_2+i\Omega}{2}\right)}}{\sqrt{\mathrm{Det}\left(\frac{\sigma_2-\sigma_1}{2}\right)}} \tag{42}$$

Now we can consider

$$Q_\alpha(\rho,\tau) = \mathrm{Tr}\left[\rho^\alpha \tau^{1-\alpha}\right] \tag{43}$$

$$= \frac{1}{(Z_\rho)^\alpha (Z_\tau)^{1-\alpha}} \mathrm{Tr}\left[e^{-\frac{1}{2}\hat{r}^T \alpha H_\rho \hat{r}} e^{-\frac{1}{2}\hat{r}^T[-(\alpha-1)H_\tau]\hat{r}}\right] \tag{44}$$

We can apply the above relation to obtain the following, when $\sigma_{\tau(\alpha-1)} > \sigma_{\rho(\alpha)}$

$$\mathrm{Tr}\left[\rho^\alpha \tau^{1-\alpha}\right] = \frac{Z_{\rho(\alpha)} Z_{\tau(\alpha-1)}}{(Z_\rho)^\alpha (Z_\tau)^{1-\alpha}} \frac{1}{\sqrt{\mathrm{Det}\left(\frac{\sigma_{\tau(\alpha-1)}-\sigma_{\rho(\alpha)}}{2}\right)}} \tag{45}$$

where $Z_{\rho(\alpha)}$, $Z_{\tau(\alpha-1)}$, and $\sigma_{\rho(\alpha)}$ and $\sigma_{\tau(1-\alpha)}$ are defined similarly. The above expression simplifies significantly for $\alpha = 2$. In that case, for $\sigma_\tau > \sigma_{\rho(2)}$

$$\mathrm{Tr}[\rho^2 \tau^{-1}] = \frac{Z_{\rho(2)}(Z_\tau)^2}{(Z_\rho)^2} \frac{1}{\sqrt{\mathrm{Det}\left(\frac{\sigma_\tau - \sigma_{\rho(2)}}{2}\right)}} \tag{46}$$

where

$$\sigma_{\rho(2)} = \frac{1}{2}\left(\sigma_\rho + \Omega \sigma_\rho^{-1} \Omega^T\right) \tag{47}$$

2.3 Sandwiched Petz-Rényi relative entropy

Let's now consider the sandwiched Petz-Rényi relative entropy:

$$\widetilde{Q}_\alpha(\rho,\tau) = \mathrm{Tr}\left[\left(\tau^{\frac{1-\alpha}{2\alpha}} \rho \tau^{\frac{1-\alpha}{2\alpha}}\right)^\alpha\right] \tag{48}$$

$$= \mathrm{Tr}\left[\left(\rho^{\frac{1}{2}} \tau^{\frac{1-\alpha}{\alpha}} \rho^{\frac{1}{2}}\right)^\alpha\right] \tag{49}$$

For states ρ and τ as defined in Eqs. (8) and (9),

$$\widetilde{Q}_\alpha(\rho,\tau) = \frac{1}{(Z_\rho)^\alpha (Z_\tau)^{1-\alpha}} \mathrm{Tr}\left[\left(e^{-\frac{1}{2}\hat{r}^T \frac{1}{2} H_\rho \hat{r}} e^{-\frac{1}{2}\hat{r}^T[\beta H_\tau]\hat{r}} e^{-\frac{1}{2}\hat{r}^T \frac{1}{2} H_\rho \hat{r}}\right)^\alpha\right], \tag{50}$$

where $\beta = (1-\alpha)/\alpha$. We use the fact that

$$e^{-\frac{1}{2}\hat{r}^T \frac{1}{2} H_1 \hat{r}} e^{-\frac{1}{2}\hat{r}^T H_2 \hat{r}} e^{-\frac{1}{2}\hat{r}^T \frac{1}{2} H_1 \hat{r}} = e^{-\frac{1}{2}\hat{r}^T H_3 \hat{r}} \tag{51}$$

where (see Prop. 8, Ref. [1])

$$H_3 = 2i\Omega \mathrm{arccoth}(\sigma_3 i\Omega), \tag{52}$$

$$\sigma_3 = \sigma_1 - \left(\sqrt{I + (\sigma_1\Omega)^{-2}}\right)\sigma_1(\sigma_1+\sigma_2)^{-1}\sigma_1\left(\sqrt{I+(\Omega\sigma_1)^{-2}}\right). \tag{53}$$

Using this, we find that

$$e^{-\frac{1}{2}\hat{r}^T\frac{1}{2}H_\rho\hat{r}}e^{-\frac{1}{2}\hat{r}^T\beta H_\tau\hat{r}}e^{-\frac{1}{2}\hat{r}^T\frac{1}{2}H_\rho\hat{r}} = e^{-\frac{1}{2}\hat{r}^T H_\zeta\hat{r}}, \tag{54}$$

where

$$H_\zeta = 2i\Omega\text{arccoth}(\sigma_\zeta i\Omega), \tag{55}$$

$$\sigma_\zeta = \sigma_\rho - \left(\sqrt{I+(\sigma_\rho\Omega)^{-2}}\right)\sigma_\rho(\sigma_\rho+\sigma_{\tau(\beta)})^{-1}\sigma_\rho\left(\sqrt{I+(\Omega\sigma_\rho)^{-2}}\right), \tag{56}$$

$$\sigma_{\tau(\beta)} = \frac{\left[I+(\sigma_\tau i\Omega)^{-1}\right]^\beta + \left[I-(\sigma_\tau i\Omega)^{-1}\right]^\beta}{\left[I+(\sigma_\tau i\Omega)^{-1}\right]^\beta - \left[I-(\sigma_\tau i\Omega)^{-1}\right]^\beta} i\Omega. \tag{57}$$

Finally, we exponentiate this expression with α from the definition of sandwiched Petz-Rényi overlap. We use the fact that this scales the resultant Hamiltonian matrix by a factor α so as to obtain

$$\text{Tr}\left[e^{-\frac{1}{2}\hat{r}^T\alpha H_\zeta\hat{r}}\right] = \sqrt{\text{Det}\left(\frac{\sigma_{\zeta(\alpha)}+i\Omega}{2}\right)} \tag{58}$$

where

$$\sigma_{\zeta(\alpha)} = \frac{\left[I+(\sigma_\zeta i\Omega)^{-1}\right]^\alpha + \left[I-(\sigma_\zeta i\Omega)^{-1}\right]^\alpha}{\left[I+(\sigma_\zeta i\Omega)^{-1}\right]^\alpha - \left[I-(\sigma_\zeta i\Omega)^{-1}\right]^\alpha} i\Omega \tag{59}$$

Thus we obtain our formula for the sandwiched Petz-Rényi overlap as:

$$\widetilde{Q}_\alpha(\rho,\tau) = \frac{1}{(Z_\rho)^\alpha(Z_\tau)^{1-\alpha}}\sqrt{\text{Det}\left(\frac{\sigma_{\zeta(\alpha)}+i\Omega}{2}\right)} \tag{60}$$

Fidelity is a special case of sandwiched Petz-Rényi overlap. When $\alpha = \frac{1}{2}$, we have

$$F[\rho,\tau] = \left(\widetilde{Q}_{\alpha=\frac{1}{2}}(\rho,\tau)\right)^2 \tag{61}$$

Fidelity: If $\alpha = \frac{1}{2}$, we have $\beta = 1$, and $\tau(\beta) = \tau$. Thus we have

$$\sigma_\zeta = \sigma_\rho - \left(\sqrt{I+(\sigma_\rho\Omega)^{-2}}\right)\sigma_\rho(\sigma_\rho+\sigma_\tau)^{-1}\sigma_\rho\left(\sqrt{I+(\sigma_\rho\Omega)^{-2}}\right), \tag{62}$$

$$\sigma_{\zeta(\alpha=\frac{1}{2})} = \left(\sqrt{I+(\sigma_\zeta\Omega)^{-2}}+I\right)\sigma_\zeta, \tag{63}$$

and thus the fidelity becomes

$$F[\rho,\tau] = \text{Tr}\left[\sqrt{\rho^{\frac{1}{2}}\tau\rho^{\frac{1}{2}}}\right]^2 \tag{64}$$

$$= \frac{\text{Det}\left[\sigma_{\zeta(\frac{1}{2})}\right]}{Z_\rho Z_\tau} \tag{65}$$

Now we will find an expression for the sandwiched Petz-Rényi overlap when $\alpha > 1$. For simplicity, define $\gamma = -\beta > 0$. When $\sigma_{\tau(\gamma)} > \sigma_\rho$, we have

$$\widetilde{Q}_\alpha(\rho,\tau) = \frac{1}{(Z_\rho)^\alpha(Z_\tau)^{1-\alpha}}\sqrt{\text{Det}\left(\frac{\sigma_{\zeta(\alpha)}+i\Omega}{2}\right)} \tag{66}$$

where

$$\sigma_{\zeta(\alpha)} = \frac{\left[I + (\sigma_\zeta i\Omega)^{-1}\right]^\alpha + \left[I - (\sigma_\zeta i\Omega)^{-1}\right]^\alpha}{\left[I + (\sigma_\zeta i\Omega)^{-1}\right]^\alpha - \left[I - (\sigma_\zeta i\Omega)^{-1}\right]^\alpha} i\Omega, \tag{67}$$

$$\sigma_\zeta = \sigma_\rho + \left(\sqrt{I + (\sigma_\rho \Omega)^{-2}}\right) \sigma_\rho (\sigma_{\tau(\gamma) - \sigma_\rho})^{-1} \sigma_\rho \left(\sqrt{I + (\sigma_\rho \Omega)^{-2}}\right), \tag{68}$$

$$\sigma_{\tau(\gamma)} = \frac{\left[I + (\sigma_\tau i\Omega)^{-1}\right]^\gamma + \left[I - (\sigma_\tau i\Omega)^{-1}\right]^\gamma}{\left[I + (\sigma_\tau i\Omega)^{-1}\right]^\gamma - \left[I - (\sigma_\tau i\Omega)^{-1}\right]^\gamma} i\Omega. \tag{69}$$

References

[1] K. P. Seshadreesan, L. Lami, and M. M. Wilde, *J. Math. Phys.* **59**, 072204 (2018).