PHYS 7895: Gaussian Quantum Information

Lecture 11

Lecturer: Mark M. Wilde *Scribe: Kunal Sharma*

1 Overview

In the last lecture, we reviewed a method to find the symplectic eigenvalues of a positive definite matrix. We then derived a relation between the Hamiltonian matrix and the covariance matrix corresponding to a faithful Gaussian state. Finally, we reviewed the conditions for the purity of a Gaussian state and found an expression for the von Neumann entropy of a Gaussian state

In this lecture, we find the quantum relative entropy and the Rényi entropies for faithful Gaussian states. We point readers to [Ser17] for background on topics covered in this lecture.

2 Relative entropy of faithful Gaussian states

The quantum relative entropy $D(\rho\|\tau)$ of a density operator ρ and a positive definite operator τ is defined as follows:

$$D(\rho\|\tau) = \operatorname{Tr}\{\rho(\ln\rho - \ln\tau)\} \ . \tag{1}$$

This is the formula for the finite-dimensional case, and it turns out to be legitimate for faithful Gaussian states.

In the last lecture we showed that

$$\operatorname{Tr}\{\rho\ln\rho\} = -\frac{1}{2}\ln\operatorname{Det}[(\sigma_\rho + i\Omega)/2] - \frac{1}{4}\operatorname{Tr}\{H_\rho\sigma_\rho\} \tag{2}$$

We now calculate $-\operatorname{Tr}\{\rho\ln\tau\}$. Consider that

$$\rho = \hat{D}_{-\bar{r}_\rho}\rho_0\hat{D}_{\bar{r}_\rho}, \tag{3}$$

where ρ_0 has zero mean and the covariance matrix is σ_ρ. Then using cyclicity of trace and functional calculus of $\ln(\cdot)$, we find that

$$-\operatorname{Tr}\{\rho\ln\tau\} = -\operatorname{Tr}\{\rho_0\ln\hat{D}_{\bar{r}_\rho}\tau\hat{D}_{-\bar{r}_\rho}\}. \tag{4}$$

Let

$$\tau = \frac{\exp[-(1/2)(\hat{r}-\bar{r}_\tau)^T H_\tau(\hat{r}-\bar{r}_\tau)]}{\sqrt{\operatorname{Det}[(\sigma_\tau + i\Omega)/2]}}. \tag{5}$$

Then

$$\hat{D}_{\bar{r}_\rho}\tau\hat{D}_{-\bar{r}_\rho} = \frac{\exp[-(1/2)(\hat{r}-\delta)^T H_\tau(\hat{r}-\delta)]}{\sqrt{\mathrm{Det}[(\sigma_\tau + i\Omega)/2]}} \tag{6}$$

with $\delta = \bar{r}_\tau - \bar{r}_\rho$. Therefore,

$$\begin{aligned} &-\mathrm{Tr}\{\rho_0 \ln \hat{D}_{\bar{r}_\rho}\tau\hat{D}_{-\bar{r}_\rho}\} \\ &\quad = -\mathrm{Tr}\{\rho_0 \ln \frac{1}{\sqrt{\mathrm{Det}[(\sigma_\tau + i\Omega)/2]}}\} + \mathrm{Tr}\{\rho_0\left(\frac{1}{2}(\hat{r}-\delta)^T H_\tau(\hat{r}-\delta)\right)\} && (7) \\ &\quad = \frac{1}{2}\ln\mathrm{Det}[(\sigma_\tau + i\Omega)/2] + \frac{1}{2}\mathrm{Tr}\{\rho_0(\hat{r}-\delta)^T H_\tau(\hat{r}-\delta)\}\ . && (8) \end{aligned}$$

We now focus on the second term.

$$\begin{aligned} &\frac{1}{2}\sum_{j,k}\mathrm{Tr}\{\rho_0(\hat{r}_j-\delta_j)(\hat{r}_k-\delta_k)\}H^\tau_{j,k} \\ &\quad = \frac{1}{2}\sum_{j,k}(\mathrm{Tr}\{\rho_0\hat{r}_j\hat{r}_k\} - \mathrm{Tr}\{\rho_0\hat{r}_k\}\delta_j - \mathrm{Tr}\{\rho_0\hat{r}_j\}\delta_k + \mathrm{Tr}\{\rho_0\}\delta_j\delta_k)H^\tau_{j,k} && (9) \\ &\quad = \frac{1}{2}\sum_{j,k}(\mathrm{Tr}\{\rho_0\hat{r}_j\hat{r}_k\} + \delta_j\delta_k)H^\tau_{j,k} && (10) \\ &\quad = \frac{1}{2}\sum_{j,k}\mathrm{Tr}\{\rho_0\hat{r}_j\hat{r}_k\}H^\tau_{j,k} + \frac{1}{2}\delta^T H_\tau\delta && (11) \\ &\quad = \frac{1}{4}\mathrm{Tr}\{\sigma_\rho H_\tau\} + \frac{1}{2}\delta^T H_\tau\delta\ . && (12) \end{aligned}$$

From (8) and (12), we get

$$-\mathrm{Tr}\{\rho\ln\tau\} = \frac{1}{2}\ln\mathrm{Det}[(\sigma_\tau + i\Omega)/2] + \frac{1}{4}\mathrm{Tr}\{\sigma_\rho H_\tau\} + \frac{1}{2}\delta^T H_\tau\delta\ , \tag{13}$$

where $\delta = \bar{r}_\tau - \bar{r}_\rho$.

Therefore, the quantum relative entropy of two Gaussian states ρ and τ is given by

$$\begin{aligned} D(\rho\|\tau) &= \frac{1}{2}\left[\ln\left(\frac{\mathrm{Det}[(\sigma_\tau + i\Omega)/2]}{\mathrm{Det}[(\sigma_\rho + i\Omega)/2]}\right) + \frac{1}{2}\mathrm{Tr}\{\sigma_\rho(H_\tau - H_\rho)\} + \delta^T H_\tau\delta\right] && (14) \\ &= \frac{1}{2}\left[\ln\left(\frac{\mathrm{Det}[\sigma_\tau + i\Omega]}{\mathrm{Det}[\sigma_\rho + i\Omega]}\right) + \frac{1}{2}\mathrm{Tr}\{\sigma_\rho(H_\tau - H_\rho)\} + \delta^T H_\tau\delta\right]\ . && (15) \end{aligned}$$

The aforementioned expression is finite whenever τ is faithful.

3 Computing Rényi entropies and powers of Gaussian states

In this section, we first find Rényi entropies of Gaussian states in terms of symplectic eigenvalues. We then find the power of Gaussian states in terms of the mean vector and the covariance matrix.

3.1 Rény entropies of Gaussian states

The quantum Rényi entropy of a quantum state ρ is defined as

$$S_\alpha(\rho) = \frac{1}{1-\alpha} \ln \mathrm{Tr}\{\rho^\alpha\}, \tag{16}$$

for $\alpha \in (0,1) \cup (1,\infty)$. Moreover,

$$\lim_{\alpha \to 1} S_\alpha(\rho) = S(\rho). \tag{17}$$

Our goal is to find $\mathrm{Tr}\{\rho^\alpha\}$. Using the fact that

$$\rho = \hat{D}_{-\bar{r}} \hat{S} \left(\bigotimes_{j=1}^{n} \theta(\bar{n}_j) \right) \hat{S}^\dagger \hat{D}_{\bar{r}} \ , \tag{18}$$

we find that

$$\mathrm{Tr}\{\rho^\alpha\} = \prod_{j=1}^{n} \mathrm{Tr}\{\theta(\bar{n}_j)^\alpha\} \ . \tag{19}$$

Consider the following chain of equalities:

$$\begin{aligned} \mathrm{Tr}\{\theta(\bar{n})^\alpha\} &= \frac{1}{(\bar{n}+1)^\alpha} \sum_{n=0}^{\infty} \left(\frac{\bar{n}}{\bar{n}+1} \right)^{\alpha n} && (20) \\ &= \frac{1}{(\bar{n}+1)^\alpha} \frac{1}{1-(\bar{n}/(\bar{n}+1))^\alpha} && (21) \\ &= \frac{1}{(\bar{n}+1)^\alpha - \bar{n}^\alpha} \ , && (22) \end{aligned}$$

which implies that

$$\mathrm{Tr}\{\rho^\alpha\} = \prod_{j=1}^{n} \frac{1}{(\bar{n}_j+1)^\alpha - \bar{n}_j^\alpha}. \tag{23}$$

In terms of symplectic eigenvalues $\nu_j = 2\bar{n}_j + 1$, $\mathrm{Tr}\{\rho^\alpha\}$ is given by

$$\mathrm{Tr}\{\rho^\alpha\} = \prod_{j=1}^{n} \frac{2^\alpha}{(\nu_j+1)^\alpha - (\nu_j-1)^\alpha}. \tag{24}$$

Therefore, from (16) and (24), it follows that

$$S_\alpha(\rho) = \frac{1}{1-\alpha} \sum_{j=1}^{n} \ln \left(\frac{2^\alpha}{(\nu_j+1)^\alpha - (\nu_j-1)^\alpha} \right) \ . \tag{25}$$

The Rényi entropy can also be expressed as

$$S_\alpha(\rho) = \frac{\alpha}{1-\alpha} \ln \mathrm{Tr}\{\rho^\alpha\}^{1/\alpha} \tag{26}$$

$$= \frac{\alpha}{1-\alpha} \ln \|\rho\|_\alpha \ . \tag{27}$$

Therefore,

$$S_\infty(\rho) = -\ln \|\rho\|_\infty \equiv S_{\min}(\rho) \tag{28}$$

We now find $S_\infty(\rho)$ using the fact that $\|\theta(\bar{n})\|_\infty = 1/(\bar{n}+1)$. Consider the following chain of equalities:

$$S_\infty(\rho) = -\ln \left\| \bigotimes_{j=1}^{n} \theta(\bar{n}_j) \right\|_\infty \tag{29}$$

$$= -\ln \prod_{j=1}^{n} \|\theta(\bar{n}_j)\|_\infty \tag{30}$$

$$= \sum_{j=1}^{n} -\ln(1/(\bar{n}_j+1)) \tag{31}$$

$$= \sum_{j=1}^{n} \ln(\bar{n}_j+1) \tag{32}$$

$$= \sum_{j=1}^{n} \ln[(\nu_j+1)/2] \ . \tag{33}$$

In general, the following relation holds for the Rényi entropy:

$$S_\alpha(\rho) \geq S_\beta(\rho), \tag{34}$$

for $\alpha \leq \beta$.

We now find the difference between $S(\rho)$ and $S_\infty(\rho)$. Consider the following chain of inequalities:

$$S(\rho) - S_\infty(\rho) = \sum_{j=1}^{n} g(\bar{n}_j) - \ln(\bar{n}_j+1) \tag{35}$$

$$= \sum_{j=1}^{n} (\bar{n}_j+1)\ln(\bar{n}_j+1) - \bar{n}_j \ln \bar{n}_j - \ln(\bar{n}_j+1) \tag{36}$$

$$= \sum_{j=1}^{n} \ln[((\bar{n}_j+1)/\bar{n}_j)^{\bar{n}_j}] \tag{37}$$

$$\leq \sum_{j=1}^{n} \ln(e) \tag{38}$$

$$= n \ . \tag{39}$$

Therefore, the difference between $S(\rho)$ and $S_\infty(\rho)$ never exceeds the number of modes.

3.2 Power of Gaussian state

Let

$$\rho = \frac{1}{\sqrt{\mathrm{Det}[(\sigma + i\Omega)/2)]}} \exp\left(-\frac{1}{2}(\hat{r} - \bar{r})^T H(\hat{r} - \bar{r})\right) \tag{40}$$

$$= \hat{D}_{-\bar{r}} \left[\frac{\exp\left(-\frac{1}{2}\hat{r}^T H \hat{r}\right)}{\sqrt{\mathrm{Det}[(\sigma + i\Omega)/2)]}}\right] \hat{D}_{\bar{r}} \ . \tag{41}$$

Therefore,

$$\rho^\alpha \propto \hat{D}_{-\bar{r}} \exp(-(1/2)\hat{r}^T \alpha H \hat{r}) \hat{D}_{\bar{r}}. \tag{42}$$

Let $H_{(\alpha)} = \alpha H$. Then there exists a corresponding $\sigma_{(\alpha)}$ such that

$$\frac{\rho^\alpha}{\mathrm{Tr}\{\rho^\alpha\}} = \frac{\hat{D}_{-\bar{r}} \exp(-(1/2)\hat{r}^T H_{(\alpha)} \hat{r}) \hat{D}_{\bar{r}}}{\sqrt{\mathrm{Det}[(\sigma_{(\alpha)} + i\Omega)/2]}} \tag{43}$$

$$= \frac{\exp(-(1/2)(\hat{r} - \bar{r})^T H_{(\alpha)} (\hat{r} - \bar{r}))}{\sqrt{\mathrm{Det}[(\sigma_{(\alpha)} + i\Omega)/2]}} \ . \tag{44}$$

To determine $\sigma_{(\alpha)}$ in terms of σ, we use the following formulas derived in the previous lecture:

$$\sigma = \coth(i\Omega H/2) i\Omega, \tag{45}$$

$$H = 2\,\mathrm{arccoth}(i\Omega\sigma) i\Omega \ . \tag{46}$$

Consider that

$$\sigma_{(\alpha)} = \coth(i\Omega H_{(\alpha)}/2) i\Omega \tag{47}$$

$$= \coth(i\Omega \alpha H/2) i\Omega \tag{48}$$

$$= \coth(i\Omega\alpha/2[2\,\mathrm{arccoth}(\sigma i\Omega) i\Omega]) i\Omega \tag{49}$$

$$= \coth(\alpha\,\mathrm{arccoth}(\sigma i\Omega)) i\Omega. \tag{50}$$

For $|x| > 1$, we have that

$$\coth(\alpha\,\mathrm{arccoth}(x)) = \frac{(1 + 1/x)^\alpha + (1 - 1/x)^\alpha}{(1 + 1/x)^\alpha - (1 - 1/x)^\alpha} \ . \tag{51}$$

Since eigenvalues of $\sigma i\Omega$ are either greater than 1 or less than -1, by using (51) we find that

$$\sigma_{(\alpha)} = \frac{(I + (\sigma i\Omega)^{-1})^\alpha + (I - (\sigma i\Omega)^{-1})^\alpha}{(I + (\sigma i\Omega)^{-1})^\alpha - (I - (\sigma i\Omega)^{-1})^\alpha} i\Omega \ , \tag{52}$$

which implies that

$$\frac{\rho^\alpha}{\mathrm{Tr}\{\rho^\alpha\}} = \frac{\exp(-(1/2)(\hat{r} - \bar{r})^T H_{(\alpha)} (\hat{r} - \bar{r}))}{\sqrt{\mathrm{Det}[(\sigma_{(\alpha)} + i\Omega)/2]}} \ . \tag{53}$$

Moreover,

$$\mathrm{Tr}\{\rho^\alpha\} = \mathrm{Tr}\left\{\left(\frac{\exp(-(1/2)(\hat{r}-\bar{r})^T H(\hat{r}-\bar{r}))}{\sqrt{\mathrm{Det}[(\sigma+i\Omega)/2]}}\right)^\alpha\right\} \tag{54}$$

$$= \frac{1}{(\mathrm{Det}[(\sigma+i\Omega)/2)])^{\alpha/2}}\,\mathrm{Tr}\{\exp(-(1/2)(\hat{r}-\bar{r})^T \alpha H(\hat{r}-\bar{r}))\} \tag{55}$$

$$= \frac{1}{(\mathrm{Det}[(\sigma+i\Omega)/2])^{\alpha/2}}\sqrt{\mathrm{Det}[(\sigma_{(\alpha)}+i\Omega)/2]}\ . \tag{56}$$

We now focus on two special cases: $\alpha = 2$ and $\alpha = 1/2$. For $\alpha = 2$, we have

$$\frac{(1+1/x)^\alpha + (1-1/x)^\alpha}{(1+1/x)^\alpha - (1-1/x)^\alpha} = \frac{1}{2}(x+x^{-1}) \tag{57}$$

Therefore,

$$\sigma_{(2)} = \frac{1}{2}(\sigma i\Omega + (\sigma i\Omega)^{-1})i\Omega \tag{58}$$

$$= \frac{1}{2}(\sigma + i\Omega\sigma^{-1}i\Omega) \tag{59}$$

$$= \frac{1}{2}(\sigma + \Omega\sigma^{-1}\Omega^T)\ , \tag{60}$$

which implies that

$$\frac{\rho^2}{\mathrm{Tr}\{\rho^2\}} = \frac{\exp(-(1/2)(\hat{r}-\bar{r})^T 2H(\hat{r}-\bar{r}))}{\sqrt{\mathrm{Det}[((1/2)(\sigma+\Omega\sigma^{-1}\Omega^T)+i\Omega)/2]}}\ . \tag{61}$$

Moreover, the purity of the Gaussian state ρ is given by

$$\mathrm{Tr}\{\rho^2\} = \frac{1}{\mathrm{Det}[(\sigma+i\Omega)/2]}\sqrt{\mathrm{Det}[((1/2)(\sigma+\Omega\sigma^{-1}\Omega^T)+i\Omega)/2]}\ , \tag{62}$$

which further reduces (after many steps) to

$$\mathrm{Tr}\{\rho^2\} = \frac{1}{\sqrt{\mathrm{Det}(\sigma)}}\ . \tag{63}$$

We note that the same expression for the purity of a Gaussian state was derived in the previous lecture by using a different approach.

Let $\alpha = 1/2$. Consider that

$$\frac{\rho^{1/2}}{\mathrm{Tr}\{\rho^{1/2}\}} = \frac{\exp(-(1/2)(\hat{r}-\bar{r})^T H_{(1/2)}(\hat{r}-\bar{r}))}{\sqrt{\mathrm{Det}[(\sigma_{(1/2)}+i\Omega)/2]}}, \tag{64}$$

where $H_{(1/2)} = 1/2H$. Moreover, for $\alpha = 1/2$, we have

$$\frac{(1+1/x)^\alpha + (1-1/x)^\alpha}{(1+1/x)^\alpha - (1-1/x)^\alpha} = (1+\sqrt{1-1/x^2})x\ , \tag{65}$$

which implies that

$$\sigma_{(1/2)} = (I + \sqrt{I - (\sigma i\Omega)^{-2}})(\sigma i\Omega)i\Omega \tag{66}$$

$$= (\sqrt{I + (\sigma\Omega)^{-2}} + I)\sigma \ . \tag{67}$$

Therefore,

$$\mathrm{Tr}\{\rho^{1/2}\} = \frac{1}{(\mathrm{Det}[(\sigma + i\Omega)/2])^{1/4}} \sqrt{\mathrm{Det}[((\sqrt{I + (\sigma\Omega)^{-2}} + I)\sigma + i\Omega)/2]} \ . \tag{68}$$

References

[Ser17] Alessio Serafini. *Quantum Continuous Variables: A Primer of Theoretical Methods*. CRC Press, 2017.