PHYS 7895: Gaussian Quantum Information

Lecture 8

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1 Overview

In the previous lecture, we studied transformations of quantum states under evolutions induced by both linear and quadratic Hamiltonians.

In this lecture, we will continue on the same track and proceed to define faithful Gaussian states. Further, we will discuss the most general form that a Gaussian state can take.

2 Quadratic Hamiltonians

2.1 Faithful quantum states

Consider a Hamiltonian of the form

$$\hat{H} = \frac{1}{2}\hat{r}^T H \hat{r} + \hat{r}^T \overline{r}' \tag{1}$$

where $\overline{r}' \in \mathbb{R}^{2n}$ and H is a positive definite $2n \times 2n$ real matrix. A faithful n-mode Gaussian state is defined as follows:

$$\frac{e^{-\beta \hat{H}}}{\operatorname{Tr}\left[e^{-\beta \hat{H}}\right]} \text{ for } \beta > 0. \tag{2}$$

The word faithful means that the state is positive definite, which also means that it has full support.

Consider that

$$\hat{H}' \equiv \frac{1}{2}(\hat{r} - \overline{r})^T H (\hat{r} - \overline{r}) \tag{3}$$

$$= \frac{1}{2}\left(\hat{r}^T H \hat{r} - 2\overline{r}^T H \hat{r} + \|\overline{r}\|_2^2\right) \tag{4}$$

$$= \frac{1}{2}\hat{r}^T H \hat{r} - \hat{r}^T H \overline{r} + \frac{1}{2}\|\overline{r}\|_2^2. \tag{5}$$

Now if we set $\overline{r} = -H^{-1}\overline{r}$, we recover the original form of the Hamiltonian in (1) up to an additive constant. That constant term $\|\overline{r}\|_2^2$ can be eliminated after normalization. Furthermore, β can be subsumed into H.

Thus, we take our formal definition of faithful Gaussian states to be as follows:

Definition 1. *A **faithful** n**-mode Gaussian state** is defined as follows:*

$$\frac{\exp\left(-\frac{1}{2}(\hat{r}-\overline{r})^T H(\hat{r}-\overline{r})\right)}{\mathrm{Tr}\left[\exp\left(-\frac{1}{2}(\hat{r}-\overline{r})^T H(\hat{r}-\overline{r})\right)\right]}. \tag{6}$$

where $\overline{r} \in \mathbb{R}^{2n}$ and H is a positive definite $2n \times 2n$ real matrix.

It is natural at this point to consider computing the mean vector, covariance matrix, and normalization for a faithful Gaussian state parameterized by $\overline{r}$ and H.

2.2 Simple example of a single-mode state

We will start with perhaps the most simple example possible. Consider a single-mode state with Hamiltonian matrix

$$H = \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{7}$$

$\lambda > 0$, and $\overline{r} = 0$.

Then the state is given by

$$\rho = \frac{e^{-\frac{1}{2}\hat{r}^T H \hat{r}}}{\mathrm{Tr}\left[e^{-\frac{1}{2}\hat{r}^T H \hat{r}}\right]}. \tag{8}$$

Consider that

$$\frac{1}{2}\hat{r}^T H \hat{r} = \frac{1}{2}\begin{pmatrix} \hat{x} & \hat{p} \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{p} \end{pmatrix} \tag{9}$$

$$= \frac{\lambda}{2}(\hat{x}^2 + \hat{p}^2). \tag{10}$$

If we now use

$$\hat{n} = \hat{a}^\dagger \hat{a} = \left(\frac{\hat{x} - i\hat{p}}{\sqrt{2}}\right)\left(\frac{\hat{x} + i\hat{p}}{\sqrt{2}}\right) \tag{11}$$

$$= \frac{1}{2}\left(\hat{x}^2 + \hat{p}^2 + i[\hat{x}, \hat{p}]\right) \tag{12}$$

$$= \frac{1}{2}\left(\hat{x}^2 + \hat{p}^2 - 1\right), \tag{13}$$

then

$$\frac{1}{2}\hat{r}^T H \hat{r} = \lambda(\hat{n} + 1/2). \tag{14}$$

We can use the fact that $\hat{n} = \sum_{n=0}^{\infty} n|n\rangle\langle n|$ to write

$$e^{-\frac{1}{2}\hat{r}^T H\hat{r}} = \sum_{n=0}^{\infty} e^{-\lambda(n+\frac{1}{2})}|n\rangle\langle n| \tag{15}$$

$$= e^{-\frac{\lambda}{2}} \sum_{n=0}^{\infty} e^{-\lambda n}|n\rangle\langle n| \tag{16}$$

$$\implies \mathrm{Tr}\big[e^{-\frac{1}{2}\hat{r}^T H\hat{r}}\big] = e^{-\frac{\lambda}{2}} \sum_{n=0}^{\infty} e^{-\lambda n} \tag{17}$$

$$= e^{-\frac{\lambda}{2}} \frac{1}{1-e^{-\lambda}} \tag{18}$$

$$= \frac{1}{e^{\frac{\lambda}{2}} - e^{-\frac{\lambda}{2}}} \tag{19}$$

$$\equiv z(\lambda). \tag{20}$$

We denote the final quantity as $z(\lambda)$ due to its role as the partition function from statistical mechanics.

2.2.1 Mean vector

We now prove that any state diagonal in the Fock basis has mean vector equal to zero. This follows because

$$\langle n|\hat{x}|n\rangle = \frac{1}{\sqrt{2}} \langle n|\hat{a} + \hat{a}^\dagger|n\rangle \tag{21}$$

$$= \frac{1}{\sqrt{2}} \big[\langle n|\hat{a}|n\rangle + \langle n|\hat{a}^\dagger|n\rangle \big] \tag{22}$$

$$= \frac{1}{\sqrt{2}} \big[\sqrt{n}\, \langle n|n-1\rangle + \sqrt{n+1}\, \langle n|n+1\rangle \big] \tag{23}$$

$$= 0. \tag{24}$$

By a similar calculation, $\langle n|\hat{p}|n\rangle = 0$.

Therefore any state that is diagonal in the Fock (number state) basis has mean vector equal to zero and we can write

$$\mathrm{Tr}\big[e^{-\frac{1}{2}\hat{r}^T H\hat{r}}\hat{x}\big] = 0 = \mathrm{Tr}\big[e^{-\frac{1}{2}\hat{r}^T H\hat{r}}\hat{p}\big]. \tag{25}$$

2.2.2 Covariance matrix

It is simple to show that $\langle n|\hat{x}\hat{p} + \hat{p}\hat{x}|n\rangle = 0$ and also that

$$\langle n|2\hat{x}^2|n\rangle = 2n + 1 = \langle n|2\hat{p}^2|n\rangle\,. \tag{26}$$

It follows from

$$\hat{x}\hat{p}+\hat{p}\hat{x}=\frac{1}{2i}\left[\left(\hat{a}+\hat{a}^{\dagger}\right)\left(\hat{a}-\hat{a}^{\dagger}\right)+\left(\hat{a}-\hat{a}^{\dagger}\right)\left(\hat{a}+\hat{a}^{\dagger}\right)\right] \tag{27}$$

$$=\frac{1}{2i}\left[\hat{a}^{2}+\hat{a}^{\dagger}\hat{a}-\hat{a}\hat{a}^{\dagger}-\left(\hat{a}^{\dagger}\right)^{2}+\hat{a}^{2}-\hat{a}^{\dagger}\hat{a}+\hat{a}\hat{a}^{\dagger}-\left(\hat{a}^{\dagger}\right)^{2}\right] \tag{28}$$

$$=\frac{1}{i}\left[\hat{a}^{2}-\left(\hat{a}^{\dagger}\right)^{2}\right]. \tag{29}$$

Then we find that

$$\langle n|\left[\hat{x}\hat{p}+\hat{p}\hat{x}\right]|n\rangle=\frac{1}{i}\langle n|\left[\hat{a}^{2}-\left(\hat{a}^{\dagger}\right)^{2}\right]|n\rangle \tag{30}$$

$$=\frac{1}{i}\langle n|\hat{a}^{2}|n\rangle-\langle n|\left(\hat{a}^{\dagger}\right)^{2}|n\rangle \tag{31}$$

$$=\frac{1}{i}\sqrt{n\left(n-1\right)}\langle n|n-2\rangle-\sqrt{\left(n+1\right)\left(n+2\right)}\langle n|n+2\rangle \tag{32}$$

$$=0. \tag{33}$$

Also, we find that

$$\langle n|2\hat{x}^{2}|n\rangle=2\langle n|\left(\frac{\hat{a}+\hat{a}^{\dagger}}{\sqrt{2}}\right)^{2}|n\rangle \tag{34}$$

$$=\langle n|\hat{a}^{2}+\hat{a}^{\dagger}\hat{a}+\hat{a}\hat{a}^{\dagger}+\left(\hat{a}^{\dagger}\right)^{2}|n\rangle \tag{35}$$

$$=\langle n|\hat{a}^{2}+2\hat{a}^{\dagger}\hat{a}+I+\left(\hat{a}^{\dagger}\right)^{2}|n\rangle \tag{36}$$

$$=2n+1, \tag{37}$$

and similarly,

$$\langle n|2\hat{p}^{2}|n\rangle=2\langle n|\left(\frac{\hat{a}-\hat{a}^{\dagger}}{\sqrt{2}i}\right)^{2}|n\rangle \tag{38}$$

$$=-\langle n|\hat{a}^{2}-\hat{a}^{\dagger}\hat{a}-\hat{a}\hat{a}^{\dagger}+\left(\hat{a}^{\dagger}\right)^{2}|n\rangle \tag{39}$$

$$=\langle n|-\hat{a}^{2}+2\hat{a}^{\dagger}\hat{a}+I-\left(\hat{a}^{\dagger}\right)^{2}|n\rangle \tag{40}$$

$$=2n+1. \tag{41}$$

This means that

$$\mathrm{Tr}\left[\{\hat{x},\hat{p}\}\frac{e^{-\frac{1}{2}\hat{r}^{T}H\hat{r}}}{z(\lambda)}\right]=0 \tag{42}$$

and

$$2\mathrm{Tr}\left[\hat{x}^2\frac{e^{-\frac{1}{2}\hat{r}^T H\hat{r}}}{z(\lambda)}\right] = \frac{1}{z(\lambda)}\mathrm{Tr}\left[2\hat{x}^2\sum_{n=0}^{\infty}e^{-\lambda(n+\frac{1}{2})}|n\rangle\langle n|\right] \tag{43}$$

$$= \frac{1}{z(\lambda)}\sum_{n=0}^{\infty}e^{-\lambda(n+\frac{1}{2})}2\mathrm{Tr}\left[\hat{x}^2|n\rangle\langle n|\right] \tag{44}$$

$$= \frac{1}{z(\lambda)}\sum_{n=0}^{\infty}e^{-\lambda(n+\frac{1}{2})}(2n+1) \tag{45}$$

$$= 1 + 2\frac{e^{-\frac{\lambda}{2}}}{z(\lambda)}\sum_{n=0}^{\infty}e^{-\lambda n}n \tag{46}$$

$$= 1 + 2\frac{e^{-\frac{\lambda}{2}}}{z(\lambda)}\left[-\frac{d}{d\lambda}\left(\sum_{n=0}^{\infty}e^{-\lambda n}\right)\right] \tag{47}$$

$$= 1 + 2\frac{e^{-\frac{\lambda}{2}}}{z(\lambda)}\left[-\frac{d}{d\lambda}\left(\frac{1}{1-e^{-\lambda}}\right)\right] \tag{48}$$

$$= 1 + 2\left(1-e^{-\lambda}\right)\left[\frac{e^{-\lambda}}{(1-e^{-\lambda})^2}\right] \tag{49}$$

$$= \coth\left(\frac{\lambda}{2}\right) \tag{50}$$

$$\equiv \nu(\lambda) > 1 \text{ for } \lambda > 0 \tag{51}$$

where $\coth x = \frac{e^x+e^{-x}}{e^x-e^{-x}}$.

Similarly, we have

$$2\mathrm{Tr}\left[\hat{p}^2\frac{e^{-\frac{1}{2}\hat{r}^T H\hat{r}}}{z(\lambda)}\right] = \coth\left(\frac{\lambda}{2}\right). \tag{52}$$

So we have seen that a single-mode state with Hamiltonian matrix $H = \begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix}$ for $\lambda > 0$ has mean vector equal to zero, and covariance matrix σ given by

$$\sigma = \begin{pmatrix}\nu(\lambda) & 0\\ 0 & \nu(\lambda)\end{pmatrix}, \tag{53}$$

where $\nu(\lambda) = \coth\left(\frac{\lambda}{2}\right)$.

2.2.3 Normalization

The normalization of this state is

$$\mathrm{Tr}\left[e^{-\frac{1}{2}\hat{r}^T H\hat{r}}\right] = z(\lambda) = \frac{1}{e^{\frac{\lambda}{2}} - e^{-\frac{\lambda}{2}}}. \tag{54}$$

If, instead, we had started by specifying covariance matrix as $\sigma = \begin{pmatrix} \nu & 0 \\ 0 & \nu \end{pmatrix}$ such that

$$\begin{pmatrix} \nu & 0 \\ 0 & \nu \end{pmatrix} + i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} > 0 \tag{55}$$

then the Hamiltonian matrix elements are given by the inverse operation

$$\lambda(\nu) = 2\text{arcoth}(\nu) \tag{56}$$

where $\text{arcoth}(x) = \frac{1}{2} \ln \left(\frac{x+1}{x-1} \right)$ when $|x| > 1$.

Then the normalization function can be written as

$$z(\lambda(\nu)) = \frac{1}{2}\sqrt{\nu^2 - 1} \tag{57}$$

$$= \frac{1}{2}\sqrt{\text{Det}\left(\begin{bmatrix} \nu & i \\ -i & \nu \end{bmatrix} \right)} \tag{58}$$

$$= \frac{1}{2}\sqrt{\text{Det}\,(\sigma + i\Omega)} \tag{59}$$

$$= \sqrt{\text{Det}\left(\frac{\sigma + i\Omega}{2} \right)}. \tag{60}$$

We see that for the form of covariance matrix chosen, the normalization $\text{Tr}\left[e^{-\frac{1}{2}\hat{r}^T H \hat{r}}\right]$ can be written in terms of it.

2.3 n-mode state with diagonal Hamiltonian matrix

Consider an n-mode state with a diagonal Hamiltonian matrix

$$H = \begin{bmatrix} \lambda_1 & & & & & & \\ & \lambda_1 & & & & & \\ & & \lambda_2 & & & 0 & \\ & & & \lambda_2 & & & \\ & & & & \ddots & & \\ & 0 & & & & \lambda_n & \\ & & & & & & \lambda_n \end{bmatrix} = \bigoplus_{j=1}^{n} \lambda_j \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{61}$$

with $\lambda_j > 0\ \forall j$.

Then the Hamiltonian operator is

$$\hat{H} = \frac{1}{2}\hat{r}^T H \hat{r} = \frac{1}{2}\sum_{j=1} n\lambda_j(\hat{x}_j^2 + \hat{p}_j^2) \tag{62}$$

so that

$$e^{-\frac{1}{2}\hat{r}^T H \hat{r}} = e^{-\frac{1}{2}\sum_{j=1}^{n}\lambda_j(\hat{x}_j^2 + \hat{p}_j^2)} \tag{63}$$

$$= \bigotimes_{j=1}^{n} e^{-\frac{\lambda_j}{2}(\hat{x}_j^2 + \hat{p}_j^2)} \tag{64}$$

We can apply our arguments from the previous calculation (specifically (21) and (26)) to conclude that the **mean vector of this state is zero**.

2.3.1 Covariance matrix

The covariance matrix is a diagonal matrix given as

$$\sigma = \bigoplus_{j=1}^{n} \nu(\lambda_j) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{65}$$

where as in the previous, $\nu(\lambda_j) = \coth\left(\frac{\lambda_j}{2}\right) > 1$.

If the covariance matrix elements are given as in (65), then the Hamiltonian is $H = \bigoplus_{j=1}^{n} \lambda(\nu_j) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ for $\lambda(\nu) = 2\text{arcoth}(\nu) > 0$.

2.3.2 Normalization

The normalization is given by

$$\begin{aligned} \text{Tr}\left[\bigotimes_{j=1}^{n} e^{-\frac{\lambda_j}{2}(\hat{x}_j^2+\hat{p}_j^2)}\right] &= \prod_{j=1}^{n} \text{Tr}\left[e^{-\frac{\lambda_j}{2}(\hat{x}_j^2+\hat{p}_j^2)}\right] && (66) \\ &= \prod_{j=1}^{n} z(\lambda_j) && (67) \\ &= \prod_{j=1}^{n} \frac{1}{2}\sqrt{\nu_j^2 - 1} && (68) \\ &= \prod_{j=1}^{n} \sqrt{\text{Det}\left(\frac{\sigma_j + i\Omega_1}{2}\right)} && (69) \\ &= \sqrt{\prod_{j=1}^{n} \text{Det}\left(\frac{\sigma_j + i\Omega_1}{2}\right)} && (70) \\ &= \sqrt{\text{Det}\left(\frac{\sigma + i\Omega}{2}\right)}. && (71) \end{aligned}$$

This sequence of steps utilizes the fact that $\sigma + i\Omega = \bigoplus_{j=1}^{n} \sigma_j + i\Omega_1$ where $\sigma_j = \nu_j \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. We also used the fact that $\text{Det}(A \oplus B) = \text{Det}(A)\text{Det}(B)$.

To summarize, for multimode states with Hamiltonian matrix $H = \bigoplus_{j=1}^{n} \lambda_j \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, the state given by

$$\frac{e^{-\frac{1}{2}\hat{r}^T H \hat{r}}}{\text{Tr}\left[e^{-\frac{1}{2}\hat{r}^T H \hat{r}}\right]}$$

has mean vector equal to zero, covariance matrix

$$\sigma = \bigoplus_{j=1}^{n} \nu(\lambda_j) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{72}$$

with $\nu(\lambda_j) = \coth\left(\frac{\lambda_j}{2}\right)$ and normalization

$$\prod_{j=1}^{n} z(\lambda_j) = \sqrt{\text{Det}\left(\frac{\sigma + i\Omega}{2}\right)}. \tag{73}$$

3 Towards a general Gaussian state

In this section, we work towards establishing the most general form that a Gaussian state can take. We begin with a quadratic Hamiltonian, act upon it by congruence with a symplectic matrix S, and lastly we displace the state to obtain the most general form.

Suppose now that we take such a diagonal Hamiltonian H and act on it by congruence with a symplectic matrix S to produce a new Hamiltonian matrix H'.

$$H' = S^T H S \tag{74}$$

where $S = e^{\Omega A}$ for symmetric and real A. Consider now the state

$$\rho = \frac{e^{-\frac{1}{2}\hat{r}^T H' \hat{r}}}{\text{Tr}\left[e^{-\frac{1}{2}\hat{r}^T H' \hat{r}}\right]} = \frac{e^{-\frac{1}{2}\hat{r}^T S^T H S \hat{r}}}{\text{Tr}\left[e^{-\frac{1}{2}\hat{r}^T S^T H S \hat{r}}\right]} \tag{75}$$

3.1 Mean Vector

In the following, we will show how the mean vector of ρ as defined in (75) is equal to zero.

We have

$$S\hat{r} = e^{\Omega A}\hat{r} = e^{\frac{i}{2}\hat{r}^T A \hat{r}}\hat{r}e^{-\frac{i}{2}\hat{r}^T A \hat{r}} \tag{76}$$

$$\tag{77}$$

and

$$S^{-1}\hat{r} = e^{-\Omega A}\hat{r} = e^{\frac{i}{2}\hat{r}^T (-A) \hat{r}}\hat{r}e^{-\frac{i}{2}\hat{r}^T (-A) \hat{r}} \tag{78}$$

$$= e^{-\frac{i}{2}\hat{r}^T A \hat{r}}\hat{r}e^{\frac{i}{2}\hat{r}^T A \hat{r}}. \tag{79}$$

$$\implies \frac{1}{2}\hat{r}^T S^T H S\hat{r} = \frac{1}{2}(S\hat{r})^T H S\hat{r} \tag{80}$$

$$= \frac{1}{2}\left(e^{\frac{i}{2}\hat{r}^T A\hat{r}}\hat{r}e^{-\frac{i}{2}\hat{r}^T A\hat{r}}\right)^T H \left(e^{\frac{i}{2}\hat{r}^T A\hat{r}}\hat{r}e^{-\frac{i}{2}\hat{r}^T A\hat{r}}\right) \tag{81}$$

$$= e^{\frac{i}{2}\hat{r}^T A\hat{r}}\frac{1}{2}\hat{r}^T H\hat{r}e^{-\frac{i}{2}\hat{r}^T A\hat{r}} \tag{82}$$

$$\implies e^{-\frac{1}{2}\hat{r}^T S^T H S\hat{r}} = e^{\frac{i}{2}\hat{r}^T A\hat{r}}e^{-\frac{1}{2}\hat{r}^T H\hat{r}}e^{-\frac{i}{2}\hat{r}^T A\hat{r}} \tag{83}$$

$$\tag{84}$$

$$\implies \text{mean vector of } \frac{e^{-\frac{1}{2}\hat{r}^T S^T H S\hat{r}}}{\mathrm{Tr}\left[e^{-\frac{1}{2}\hat{r}^T S^T H S\hat{r}}\right]}$$

$$= \mathrm{Tr}\left[\frac{\hat{r}e^{\frac{i}{2}\hat{r}^T A\hat{r}}e^{-\frac{1}{2}\hat{r}^T H\hat{r}}e^{-\frac{i}{2}\hat{r}^T A\hat{r}}}{\mathrm{Tr}\left[e^{-\frac{1}{2}\hat{r}^T H\hat{r}}\right]}\right] \tag{85}$$

$$= \mathrm{Tr}\left[e^{-\frac{i}{2}\hat{r}^T A\hat{r}}\hat{r}e^{\frac{i}{2}\hat{r}^T A\hat{r}}\frac{e^{-\frac{1}{2}\hat{r}^T H\hat{r}}}{\mathrm{Tr}\left[e^{-\frac{1}{2}\hat{r}^T H\hat{r}}\right]}\right] \tag{86}$$

$$= \mathrm{Tr}\left[S^{-1}\hat{r}\frac{e^{-\frac{1}{2}\hat{r}^T H\hat{r}}}{\mathrm{Tr}\left[e^{-\frac{1}{2}\hat{r}^T H\hat{r}}\right]}\right] \tag{87}$$

$$= S^{-1}.0 = 0. \tag{88}$$

In the above, the second equality arises due to cyclicity of the trace. The third equality follows from (79). S^{-1} can then be pulled out of the trace operation and the final equality follows from the earlier performed calculations in 2.2.1.

3.2 Covariance Matrix

Since the mean vector is zero, the covariance matrix is given by

$$\sigma = \mathrm{Tr}\left[\{\hat{r},\hat{r}^T\}e^{\frac{i}{2}\hat{r}^T A\hat{r}}\frac{e^{-\frac{1}{2}\hat{r}^T H\hat{r}}}{\mathrm{Tr}\left[e^{-\frac{1}{2}\hat{r}^T H\hat{r}}\right]}e^{-\frac{i}{2}\hat{r}^T A\hat{r}}\right] \tag{89}$$

$$= \mathrm{Tr}\left[e^{-\frac{i}{2}\hat{r}^T A\hat{r}}\{\hat{r},\hat{r}^T\}e^{\frac{i}{2}\hat{r}^T A\hat{r}}\frac{e^{-\frac{1}{2}\hat{r}^T H\hat{r}}}{\mathrm{Tr}\left[e^{-\frac{1}{2}\hat{r}^T H\hat{r}}\right]}\right] \tag{90}$$

$$= \mathrm{Tr}\left[\left\{S^{-1}\hat{r},(S^{-1}\hat{r})^T\right\}\frac{e^{-\frac{1}{2}\hat{r}^T H\hat{r}}}{\mathrm{Tr}\left[e^{-\frac{1}{2}\hat{r}^T H\hat{r}}\right]}\}\right] \tag{91}$$

$$= S^{-1}\mathrm{Tr}\left[\{\hat{r},\hat{r}^T\}\frac{e^{-\frac{1}{2}\hat{r}^T H\hat{r}}}{\mathrm{Tr}\left[e^{-\frac{1}{2}\hat{r}^T H\hat{r}}\right]}\right]S^{-T} \tag{92}$$

$$= S^{-1}\sigma S^{-T} \tag{93}$$

$$\equiv \sigma'. \tag{94}$$

$$\implies \sigma' = S^{-1}\left(\bigoplus_{j=1}^{n}\coth\left(\frac{\lambda_j}{2}\right)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right)S^{-T}. \tag{95}$$

In the above, the first equality arises from the definition of the covariance matrix. The second equality results from the cyclicity of trace. The third equality arises from applying (79). The fourth equality comes from recognizing that S^{-1} and S^{-T} can be taken out of the trace. Finally one can recognize the original covariance matrix and obtain the expression for σ'.

3.3 Normalization

$$\begin{aligned}
\mathrm{Tr}\left[e^{-\frac{1}{2}\hat{r}^T H' \hat{r}}\right] &= \mathrm{Tr}\left[e^{\frac{i}{2}\hat{r}^T A \hat{r}} e^{-\frac{1}{2}\hat{r}^T H \hat{r}} e^{-\frac{i}{2}\hat{r}^T A \hat{r}}\right] && (96)\\
&= \mathrm{Tr}\left[e^{-\frac{1}{2}\hat{r}^T H \hat{r}}\right] && (97)\\
&= \sqrt{\mathrm{Det}\left(\frac{\sigma + i\Omega}{2}\right)}. && (98)
\end{aligned}$$

In the above, we used the cyclicity of trace to make the simplification.

For symplectic S, we have the following properties (proved in section 4).

$$\begin{aligned}
\mathrm{Det}(S) = 1 &= \mathrm{Det}(S^{-1}) && (99)\\
&= \mathrm{Det}(S^{-T}). && (100)
\end{aligned}$$

Thus we can write

$$\begin{aligned}
\sqrt{\mathrm{Det}\left(\frac{\sigma + i\Omega}{2}\right)} &= \sqrt{\mathrm{Det}(S^{-1})\mathrm{Det}\left(\frac{\sigma + i\Omega}{2}\right)\mathrm{Det}(S^{-T})} && (101)\\
&= \sqrt{\mathrm{Det}\left(S^{-1}\left(\frac{\sigma + i\Omega}{2}\right)S^{-T}\right)} && (102)\\
&= \sqrt{\mathrm{Det}\left(\frac{\sigma' + i\Omega}{2}\right)} && (103)
\end{aligned}$$

where we used that $S\Omega S^T = \Omega$.

3.4 Displacing the state

Now suppose that we act on the new state characterized in the above by a displacement operator $\hat{D}_{\overline{r}} = \exp(i\overline{r}^T \Omega \hat{r})$.

$$\hat{D}_{-\overline{r}} e^{-\frac{1}{2}\hat{r}^T H' \hat{r}} \hat{D}_{\overline{r}} = e^{-\frac{1}{2}\left[\hat{D}_{-\overline{r}}\hat{r}^T H' \hat{r} \hat{D}_{\overline{r}}\right]} \tag{104}$$

If we write

$$\hat{r}^T H' \hat{r} = \sum_{jk} \hat{r}_j H'_{jk} \hat{r}_k \tag{105}$$

then we can see that

$$\hat{D}_{-\overline{r}}\hat{r}^T H'\hat{r}\hat{D}_{\overline{r}} = \sum_{jk} \hat{D}_{-\overline{r}}\hat{r}_j\hat{D}_{\overline{r}}H'_{jk}\hat{D_{-\overline{r}}}\hat{r}_k\hat{D}_{\overline{r}} \tag{106}$$

$$= \sum_{jk} (\hat{r}_j - \overline{r}_j)\, H'_{jk}\, (\hat{r}_j - \overline{r}_j) \tag{107}$$

which yields

$$\hat{D}_{-\overline{r}}e^{-\frac{1}{2}\hat{r}^T H'\hat{r}}\hat{D}_{\overline{r}} = e^{-\frac{1}{2}\left[\hat{D}_{-\overline{r}}\hat{r}^T H'\hat{r}\hat{D}_{\overline{r}}\right]} \tag{108}$$

$$= e^{-\frac{1}{2}(\hat{r}-\overline{r})^T H'(\hat{r}-\overline{r})}. \tag{109}$$

This implies that under this change of the new state (via a displacement operator), the mean vector translates from zero to $\overline{r}$.

The covariance matrix, on the other hand, remains unchanged because it is invariant to changes in the mean vector alone. By this observation, it also follows that the normalization of the state is unchanged.

This **faithful Gaussian state** can be written as

$$\frac{e^{-\frac{1}{2}(\hat{r}-\overline{r})^T H'(\hat{r}-\overline{r})}}{\sqrt{\mathrm{Det}\left(\frac{\sigma'+i\Omega}{2}\right)}} \tag{110}$$

where

$$H' = S^T \bigoplus_{j=1}^{n} \lambda_j \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} S \tag{111}$$

and

$$\sigma' = S^{-1} \bigoplus_{j=1}^{n} \coth\left(\frac{\lambda_j}{2}\right) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} S^{-T}. \tag{112}$$

Notice the similarity of (110) to the expression for a classical multimode Gaussian density function.

The form of the faithful Gaussian state stated above is actually the **most general form** that a faithful Gaussian quantum state can take.

By everything that we have done in the preceding pages, we can write

$$\frac{e^{-\frac{1}{2}(\hat{r}-\overline{r})^T H'(\hat{r}-\overline{r})}}{\sqrt{\mathrm{Det}\left(\frac{\sigma'+i\Omega}{2}\right)}} = \frac{\hat{D}_{-\overline{r}}\hat{S}_A e^{-\frac{1}{2}\hat{r}^T H\hat{r}}\hat{S}_A^\dagger \hat{D}_{\overline{r}}}{\mathrm{Tr}\left[e^{-\frac{1}{2}\hat{r}^T H\hat{r}}\right]} \tag{113}$$

where

$$H = \bigoplus_{j=1}^{n} \lambda_j \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{114}$$

with $\lambda_j > 0$,

$$\hat{S}_A = e^{\frac{i}{2}\hat{r}^T A\hat{r}} \tag{115}$$

and

$$\text{Tr}\left[e^{-\frac{1}{2}\hat{r}^T H \hat{r}}\right] = \sqrt{\text{Det}\left(\frac{\sigma + i\Omega}{2}\right)} \tag{116}$$

$$= \sqrt{\text{Det}\left(\frac{\sigma' + i\Omega}{2}\right)} \tag{117}$$

with $\sigma = \bigoplus_{j=1}^{n} \nu(\lambda_j) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ where $\nu(\lambda) = \coth\left(\frac{\lambda}{2}\right)$.

4 Determinant of a symplectic matrix

In the following we prove that the determinant of a symplectic matrix is equal to one.

Lemma 2. *Any symplectic matrix has determinant equal to one.*

Proof. Consider that S is a symplectic matrix. We then have $S\Omega S^T = \Omega$. Beginning with that and taking determinant on both sides, we have the following:

$$\implies \text{Det}(S\Omega S^T) = \text{Det}(\Omega) \tag{118}$$

$$\implies \text{Det}(S)\text{Det}(\Omega)\text{Det}(S^T) = \text{Det}(\Omega) = 1 \tag{119}$$

$$\implies \text{Det}(S)\text{Det}(S^T) = 1 \tag{120}$$

$$\implies \text{Det}(S)^2 = 1 \tag{121}$$

$$\implies \text{Det}(S) = \pm 1. \tag{122}$$

The second line follows from the fact that $\text{Det}(\Omega) = 1$. The fourth line is due to the invariance of the determinant to transposition of its argument.

Now that we have established that $\text{Det}(S) = \pm 1$, we need to eliminate the possibility that $\text{Det}(S) = -1$ to conclude the proof.

Using the fact that any symplectic matrix is invertible (and thus full-rank), it follows that $S^T S$ is a symmetric positive definite matrix. This implies that the eigenvalues of $S^T S + \mathbb{I}$ are greater than one.

Thus

$$S^T S + \mathbb{I} = S^T(S + S^{-T}) \tag{123}$$

$$= S^T(S + \Omega S \Omega^T) \tag{124}$$

which is due to

$$S\Omega S^T = \Omega \tag{125}$$

$$\implies S\Omega S^T \Omega^T = \Omega\Omega^T = \mathbb{I} \tag{126}$$

$$\implies S^{-1} = \Omega S^T \Omega^T \tag{127}$$

$$\implies S^{-T} = \Omega S \Omega^T. \tag{128}$$

Consider that $\Omega = \bigoplus_{j=1}^{n} \Omega_1 = \mathbb{I} \otimes \Omega_1$. Then, if we write S as follows,

$$S = \sum_{j,k\in\{0,1\}} S_{jk} \otimes |j\rangle\langle k| \tag{129}$$

we get

$$S + \Omega S \Omega^T = \sum_{jk} S_{jk} \otimes |j\rangle\langle k| + (\mathbb{I} \otimes \Omega_1)\left(S_{jk} \otimes |j\rangle\langle k|\right)\left(\mathbb{I}_n \otimes \Omega_1^T\right) \tag{130}$$

$$= \sum_{jk} S_{jk} \otimes \left[\, |j\rangle\langle k| + \Omega_1 \, |j\rangle\langle k| \, \Omega_1^T \right]. \tag{131}$$

Using $\Omega_1 |0\rangle = -|1\rangle$ and $\Omega_1 |1\rangle = |0\rangle$,

$$|j\rangle\langle k| + \Omega_1 \, |j\rangle\langle k| \, \Omega_1^T = |j\rangle\langle k| + (-1)^{j+1}(-1)^{k+1} \, |j \oplus 1\rangle\langle k \oplus 1| \tag{132}$$

$$= |j\rangle\langle k| + (-1)^{j+k} \, |j \oplus 1\rangle\langle k \oplus 1| \tag{133}$$

$$\implies S + \Omega S \Omega^T = \sum_{jk} S_{jk} \otimes \left[\, |j\rangle\langle k| + \Omega_1 \, |j\rangle\langle k| \, \Omega_1^T \right] \tag{134}$$

$$= (S_{00} + S_{11}) \otimes |0\rangle\langle 0| + (S_{01} - S_{10}) \otimes |0\rangle\langle 1| \tag{135}$$

$$+ (-S_{01} + S_{10}) \otimes |1\rangle\langle 0| + (S_{00} + S_{11}) \otimes |1\rangle\langle 1| \,. \tag{136}$$

Define real matrices C and D as follows:

$$C = S_{00} + S_{11} \tag{137}$$

$$D = S_{01} - S_{10}. \tag{138}$$

$$\implies S + \Omega S \Omega^T = C \otimes |0\rangle\langle 0| + D \otimes |0\rangle\langle 1| - D \otimes |1\rangle\langle 0| + C \otimes |1\rangle\langle 1| \tag{139}$$

$$= (\mathbb{I} \otimes \underline{u})\left([C + iD] \otimes |0\rangle\langle 0| + [C - iD] \otimes |1\rangle\langle 1|\right)(\mathbb{I} \otimes \underline{u}^\dagger) \tag{140}$$

where $\underline{u} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}$.

We then get

$$0 < 1 < \mathrm{Det}(S^T S + \mathbb{I}) \tag{141}$$

$$= \mathrm{Det}\left(S^T (S + \Omega S \Omega^T)\right) \tag{142}$$

$$= \mathrm{Det}(S^T)\mathrm{Det}(S + \Omega S \Omega^T) \tag{143}$$

$$= \mathrm{Det}(S)\mathrm{Det}(\mathbb{I} \otimes \underline{u})\mathrm{Det}(C + iD)\mathrm{Det}(C - iD)\mathrm{Det}(\mathbb{I} \otimes \underline{u}^\dagger) \tag{144}$$

$$= \mathrm{Det}(S)\mathrm{Det}(C + iD)\mathrm{Det}(\overline{C + iD}) \tag{145}$$

$$= \mathrm{Det}(S)\mathrm{Det}(C + iD)\overline{\mathrm{Det}(C + iD)} \tag{146}$$

$$= \mathrm{Det}(S)|\mathrm{Det}(C + iD)|^2. \tag{147}$$

Since $\text{Det}(S)|\text{Det}(C + iD)|^2 > 0$, it must be the case that $\text{Det}(S) > 0$.

Thus we can conclude that $\text{Det}(S) \neq -1$ and hence the only remaining possibility, by necessity, is that $\text{Det}(S) = 1$. $\square$