**PHYS 7895: Gaussian Quantum Information**

Lecture 7

*Lecturer: Mark M. Wilde* *Scribe: Vishal Katariya*



# 1 Overview

In the previous lecture, we considered finite-energy states, defined the covariance matrix, and studied constraints that the covariance matrix should satisfy. Lastly, we also stated and proved the uncertainty principle for bosonic quantum states.

In this lecture, we consider generic transformations of quantum states. We study the effects of these transformations on the mean vector and covariance matrix of the state in consideration. We will define the unitary displacement operator and study its properties. Lastly, we will define another set of Hamiltonians that are quadratic in the quadrature operators, which generate another class of transformations. The study of those transformations will be completed in the following lectures.

# 2 Generic transformations of quantum states

In this section, we study generic transformations on quantum states and their effect on the mean vector and covariance matrix of the states.

## 2.1 The Displacement Operator

Suppose that we would like to shift the mean vector of an $n$-mode quantum state by a vector $\overline{r} \in \mathbb{R}^{2n}$. To do so, we define the **unitary displacement operator**.

**Definition 1.** *The unitary displacement operator $\hat{D}_{\overline{r}}$ is defined as*

$$\hat{D}_{\overline{r}} = e^{i\overline{r}^T \Omega \hat{r}} \tag{1}$$

*where $\hat{r}$ is the vector of quadrature operators as defined in earlier lectures, and $\Omega$ is the symplectic form that captures the canonical commutation relations between the quadrature operators.*

**Lemma 2.** *The displacement of an $n$-mode state is a tensor product of single-mode displacements.*

$$\hat{D}_{\overline{r}} = \hat{D}_{\overline{r}_1} \otimes \hat{D}_{\overline{r}_2} \otimes ... \otimes \hat{D}_{\overline{r}_n} \tag{2}$$

*with $\overline{r}_j = (\overline{x}_j \ \overline{p}_j)^T$.*

*Proof.* Note that

$$\overline{r}^T \Omega \hat{r} = \sum_{j,k=1}^{2n} \overline{r}_j \Omega_{jk} \hat{r}_k \tag{3}$$

$$= \sum_{j=1}^{n} \left( \overline{x}_j \hat{p}_j - \overline{p}_j \hat{x}_j \right). \tag{4}$$

Due to this component-wise expression, we have that

$$e^{i\overline{r}^T \Omega \hat{r}} = \exp\left( i \sum_{j=1}^{n} \left( \overline{x}_j \hat{p}_j - \overline{p}_j \hat{x}_j \right) \right) \tag{5}$$

and thus that

$$\hat{D}_{\overline{r}} = \hat{D}_{\overline{r}_1} \otimes \hat{D}_{\overline{r}_2} \otimes ... \otimes \hat{D}_{\overline{r}_n}, \tag{6}$$

concluding the proof. □

Further, we can think of $\overline{r}^T \Omega \hat{r}$ as a Hamiltonian. From the above analysis we note that

$$\left( \overline{r}^T \Omega \hat{r} \right)^\dagger = \overline{r}^T \Omega \hat{r}. \tag{7}$$

Since $\overline{r}^T \Omega \hat{r}$ is a Hamiltonian, it follows that $e^{i\overline{r}^T \Omega \hat{r}}$ is indeed unitary, as stated in Definition 1.

## 2.2 Inverse of the displacement operator

Observe that

$$\hat{D}_{\overline{r}}^\dagger = \left( e^{i\overline{r}^T \Omega \hat{r}} \right)^\dagger = e^{-i\overline{r}^T \Omega \hat{r}} = \hat{D}_{-\overline{r}}. \tag{8}$$

This implies that the displacement can be inverted by displacing in the opposite way.

## 2.3 Commutation Relations between Displacement Operators

Suppose that we have two displacement operators $\hat{D}_{r_1}$ and $\hat{D}_{r_2}$ for $r_1, r_2 \in \mathbb{R}^{2n}$. What is the commutation relation between the two displacement operators? In what follows, we prove the following equation:

$$\hat{D}_{r_1 + r_2} = \hat{D}_{r_1} \hat{D}_{r_2} e^{i r_1^T \Omega r_2 / 2}. \tag{9}$$

To prove this, we first need the following result.

**Lemma 3.** *When $X$ and $Y$ commute with $[X, Y]$, the following equality holds*

$$e^X e^Y = e^{X + Y + \frac{1}{2}[X,Y]}. \tag{10}$$

*Proof.* The starting point is the celebrated **Baker–Campbell–Hausdorff formula**

$$e^{X}Ye^{-X} = Y + [X,Y] + \frac{1}{2!}\left[X,[X,Y]\right] + \frac{1}{3!}\Big[X,\big[X,[X,Y]\big]\Big] + \ldots \tag{11}$$

In the case that $X$ and $Y$ both commute with $[X,Y]$, the above simplifies to the following for $s \in \mathbb{R}$.

$$e^{sX}Ye^{-sX} = Y + s[X,Y]. \tag{12}$$

Define $g(s) = e^{sX}e^{sY}$. By differentiating with respect to $s$, one obtains

$$\begin{aligned}
\frac{dg(s)}{ds} &= \frac{d}{ds}\Big(e^{sX}e^{sY}\Big) && (13)\\
&= Xe^{sX}e^{sY} + e^{sX}Ye^{sY} && (14)\\
&= Xg(s) + e^{sX}Ye^{-sX}e^{sX}e^{sY} && (15)\\
&= \Big(X + e^{sX}Ye^{-sX}\Big)g(s) && (16)\\
&= \Big(X + Y + s[X,Y]\Big)g(s). && (17)
\end{aligned}$$

The solution to this differential equation is

$$g(s) = e^{s(X+Y)+\frac{s^2}{2}[X,Y]} = e^{sX}e^{sY}. \tag{18}$$

$\forall s \in \mathbb{R}$ such that $X$ and $Y$ commute with $[X,Y]$. The second equality in the above follows from the definition of $g(s)$.

Set $s = 1$ to yield

$$e^{X}e^{Y} = e^{X+Y+\frac{1}{2}[X,Y]}, \tag{19}$$

concluding the proof. □

**Lemma 4.**

$$\hat{D}_{r_1+r_2} = \hat{D}_{r_1}\hat{D}_{r_2}e^{ir_1^T\Omega r_2/2}. \tag{20}$$

*Proof.* Set $X = e^{ir_1^T\Omega\hat{r}}$ and $Y = e^{ir_2^T\Omega\hat{r}}$.

We evaluate $[X,Y]$ as a first step.

$$\begin{aligned}
[X,Y] &= \Big[ir_1^T\Omega\hat{r}, ir_2^T\Omega\hat{r}\Big] && (21)\\
&= -\Big[r_1^T\Omega\hat{r}, r_2^T\Omega\hat{r}\Big] && (22)\\
&= -\Big[\sum_{jk} r_{1,j}\Omega_{jk}\hat{r}_k, \sum_{lm} r_{2,l}\Omega_{lm}\hat{r}_m\Big] && (23)\\
&= -\sum_{jklm} r_{1,j}r_{2,l}\Omega_{jk}\Omega_{lm}\Big[\hat{r}_k, \hat{r}_m\Big] && (24)\\
&= -\sum_{jklm} r_{1,j}r_{2,l}\Omega_{jk}\Omega_{lm}i\Omega_{km} && (25)
\end{aligned}$$

$$= -i \sum_{jklm} r_{1,j} \Omega_{jk} \Omega_{km} \Omega_{lm} r_{2,l} \tag{26}$$

$$= i \sum_{jklm} r_{1,j} \Omega_{jk} \Omega_{km} \Omega_{ml} r_{2,l} \tag{27}$$

$$= i r_1^T \Omega \Omega \Omega r_2 \tag{28}$$

$$= -i r_1^T \Omega r_2. \tag{29}$$

In the above, the first four equalities follow from algebraic manipulation and expanding $X$ and $Y$ in terms of their components. The fifth equality follows from application of the canonical commutation relation. The sixth equality again is algebraic manipulation, and the seventh equality follows from $\Omega$ being antisymmetric. The eighth equality follows from observing the expression to describe matrix multiplication. The final equality follows from the involutory nature of $i\Omega$, i.e. $\Omega^2 = -I$.

$[X, Y] = -i r_1^T \Omega r_2$ is a scalar, and hence commutes with both $X$ and $Y$. Because of this, we can apply the above Lemma 3.

$$\hat{D}_{r_1} \hat{D}_{r_2} = e^{i r_1^T \Omega \hat{r}} e^{i r_2^T \Omega \hat{r}} \tag{30}$$

$$= e^{\left(i r_1^T \Omega \hat{r} + i r_2^T \Omega \hat{r} - i/2 r_1^T \Omega r_2\right)} \tag{31}$$

$$= e^{i(r_1 + r_2)^T \Omega \hat{r}} e^{-\frac{i}{2} r_1^T \Omega r_2} \tag{32}$$

$$= \hat{D}_{r_1 + r_2} e^{-\frac{i}{2} r_1^T \Omega r_2}. \tag{33}$$

We can now write

$$\hat{D}_{r_1 + r_2} = \hat{D}_{r_1} \hat{D}_{r_2} e^{\frac{i}{2} r_1^T \Omega r_2} \tag{34}$$

which completes the proof. □

**Corollary 5.**

$$\hat{D}_{r_1} \hat{D}_{r_2} = \hat{D}_{r_2} \hat{D}_{r_1} e^{-i r_1^T \Omega r_2} \tag{35}$$

*Proof.* Apply Lemma 4 twice to get

$$\hat{D}_{r_1} \hat{D}_{r_2} e^{\frac{i}{2} r_1^T \Omega r_2} = \hat{D}_{r_1 + r_2} \tag{36}$$

$$= \hat{D}_{r_2} \hat{D}_{r_1} e^{\frac{i}{2} r_2^T \Omega r_1} \tag{37}$$

$$= \hat{D}_{r_2} \hat{D}_{r_1} e^{\frac{i}{2} r_1^T \Omega^T r_2} \tag{38}$$

$$= \hat{D}_{r_2} \hat{D}_{r_1} e^{-\frac{i}{2} r_1^T \Omega r_2}. \tag{39}$$

The first equality is a statement of Lemma 4. The second equality comes from once more applying Lemma 4. The third and fourth equalities come from considering the transpose of the argument of the exponential. □

## 2.4 Connection to traditional single-mode displacement operator

**Definition 6.** *In quantum optics, we define the single-mode displacement operator as follows:*

$$D(\alpha) \equiv \exp\left(\alpha \hat{a}^\dagger - \alpha^* \hat{a}\right) \tag{40}$$

To see its connection to the displacement operator discussed so far, consider that for $\alpha \in \mathbb{C}$ and $\alpha = \alpha_R + i\alpha_I$,

$$
\begin{aligned}
D(\alpha) &\equiv \exp\left(\alpha\hat{a}^\dagger - \alpha^*\hat{a}\right) && (41)\\
&= \exp\Big([\alpha_R + i\alpha_I]\hat{a}^\dagger - [\alpha_R - i\alpha_I]\hat{a}\Big) && (42)\\
&= \exp\Big(\alpha_R[\hat{a}^\dagger - \hat{a}] + i\alpha_I[\hat{a} + \hat{a}^\dagger]\Big) && (43)\\
&= \exp\Big(-i\sqrt{2}\alpha_R\Big[\frac{\hat{a} - \hat{a}^\dagger}{\sqrt{2}i}\Big] + i\sqrt{2}\alpha_I\Big[\frac{\hat{a} + \hat{a}^\dagger}{\sqrt{2}}\Big]\Big) && (44)\\
&= \exp\Big(i\sqrt{2}\big[\alpha_I\hat{x} - \alpha_R\hat{p}\big]\Big). && (45)
\end{aligned}
$$

It is important to keep in mind that there is a factor of $\sqrt{2}$ that must be taken care of when going between the two conventions.

To implement an $n$-mode displacement operator on a quantum state in a lab, one uses an array of highly transmissive beamsplitters and strong local oscillators in coherent states. This point will be returned to later.

## 2.5 Effect of displacement operator on mean vector of a state

**Theorem 7.** *The displacement operator $\hat{D}_{\overline{r}}$ shifts the mean vector of an arbitrary state $\rho$ by $\overline{r}$.*

$$\hat{D}^\dagger_{\overline{r}}\hat{r}\hat{D}_{\overline{r}} = \hat{r} - \overline{r}. \tag{46}$$

*Proof.* Upon action of a displacement operator $\hat{D}_{\overline{r}}$, the new mean vector of $\rho$ is given by

$$
\begin{aligned}
\overline{r}' &= \mathrm{Tr}\left[\hat{r}\hat{D}_{\overline{r}}\rho\hat{D}^\dagger_{\overline{r}}\right] && (47)\\
&= \mathrm{Tr}\left[\hat{D}^\dagger_{\overline{r}}\hat{r}\hat{D}_{\overline{r}}\rho\right] && (48)
\end{aligned}
$$

which arises from the definition of the mean vector and the cyclicity of trace.

From the above, we see that the problem of computing the new mean vector has been reduced to computing $\hat{D}^\dagger_{\overline{r}}\hat{r}\hat{D}_{\overline{r}}$, which is the same as computing

$$\hat{D}^\dagger_{\overline{r}}\hat{x}_j\hat{D}_{\overline{r}} \quad \text{and} \quad \hat{D}^\dagger_{\overline{r}}\hat{p}_j\hat{D}_{\overline{r}} \;\; \forall j \in \{1, ..., n\}. \tag{49}$$

We proved earlier in Lemma 2 that the displacement operator for $n$ modes can be written as a tensor product of single-mode displacements. This enables us to write

$$
\begin{aligned}
\hat{D}^\dagger_{\overline{r}}\hat{x}_j\hat{D}_{\overline{r}} &= \left(\hat{D}^\dagger_{\overline{r}_1} \otimes ... \otimes \hat{D}^\dagger_{\overline{r}_n}\right)\hat{x}_j\left(\hat{D}_{\overline{r}_1} \otimes ... \otimes \hat{D}_{\overline{r}_n}\right) && (50)\\
&= \hat{D}^\dagger_{\overline{r}_j}\hat{x}_j\hat{D}_{\overline{r}_j}. && (51)
\end{aligned}
$$

For $\hat{p}_j$, we can similarly write

$$\hat{D}^\dagger_{\overline{r}}\hat{p}_j\hat{D}_{\overline{r}} = \hat{D}^\dagger_{\overline{r}_j}\hat{p}_j\hat{D}_{\overline{r}_j}. \tag{52}$$

We invoke the BCH formula (Lemma 3) to calculate the above.

$$\hat{D}^\dagger_{\overline{r}_j}\hat{x}_j\hat{D}_{\overline{r}_j} = e^{i\left[\overline{p}_j\hat{x}_j - \overline{x}_j\hat{p}_j\right]}\hat{x}_j e^{-i\left[\overline{p}_j\hat{x}_j - \overline{x}_j\hat{p}_j\right]} \tag{53}$$

$$= \hat{x}_j + \Big[i\big(\overline{p}_j\hat{x}_j - \overline{x}_j\hat{p}_j\big), \hat{x}_j\Big] + \text{higher-order nested commutators that vanish} \tag{54}$$
$$= \hat{x}_j + i\Big[-\overline{x}_j\hat{p}_j, \hat{x}_j\Big] \tag{55}$$
$$= \hat{x}_j - i\overline{x}_j\big[\hat{p}_j, \hat{x}_j\big] \tag{56}$$
$$= \hat{x}_j - i\overline{x}_j(-i) \tag{57}$$
$$= \hat{x}_j - \overline{x}_j. \tag{58}$$

A similar calculation yields
$$\hat{D}^\dagger_{\overline{r}_j}\hat{p}_j\hat{D}_{\overline{r}_j} = \hat{p}_j - \overline{p}_j. \tag{59}$$

Put together, we have
$$\hat{D}^\dagger_{\overline{r}_j}\hat{r}\hat{D}_{\overline{r}_j} = \hat{r} - \overline{r}. \tag{60}$$

$$\implies \overline{r}' = \mathrm{Tr}\big[\hat{D}^\dagger_{\overline{r}_j}\hat{r}\hat{D}_{\overline{r}_j}\big] \tag{61}$$
$$= \mathrm{Tr}\big[(\hat{r} - \overline{r})\rho\big] \tag{62}$$
$$= \mathrm{Tr}\big[\hat{r}\rho\big] - \overline{r}. \tag{63}$$

where we identify $\mathrm{Tr}\big[\hat{r}\rho\big]$ as the original mean vector of $\rho$. □

### 2.6 Effect of displacement operator on covariance matrix of a state

**Lemma 8.** *The covariance matrix of a state $\rho$ is unchanged upon action by a displacement operator.*

This can be easily seen from the definition of the covariance matrix.

## 3 Quadratic Hamiltonians

The unitary displacement operator constitutes the most general evolution realizable by a Hamiltonian that is linear in the quadrature operators; i.e., the Hamiltonian is a real linear combination of the quadrature operators. It is a natural next step to examine the evolution effected by a quadratic Hamiltonian.

The general form of a quadratic Hamiltonian is as follows:
$$\hat{H} = \frac{1}{2}\hat{r}^T H\hat{r} \tag{64}$$
where $H$ is a real, symmetric $2n \times 2n$ matrix.

In the above, $\hat{H}$ is the Hamiltonian operator, and $H$ is the Hamiltonian matrix.

The Hamiltonian (64) realizes the following evolution:
$$e^{-i\hat{H}t} = e^{-\frac{i}{2}\hat{r}^T Ht\hat{r}}. \tag{65}$$

In the above, it is to be noted that the time parameter can be subsumed into the Hamiltonian matrix.

### 3.1 Effect of quadratic Hamiltonian on mean vector of a state

**Theorem 9.** *A quadratic Hamiltonian with Hamiltonian matrix $H$ changes the vector $\hat{r}$ of canonical quadrature operators as*

$$e^{i\hat{H}t}\hat{r}e^{-i\hat{H}t} = e^{\Omega H t}\hat{r}. \tag{66}$$

*Proof.* We will again invoke the BCH formula (Lemma 3) which is restated below for completeness.

$$e^{\hat{X}}\hat{Y}e^{-\hat{X}} = \hat{Y} + [\hat{X},\hat{Y}] + \frac{1}{2!}\big[\hat{X},[\hat{X},\hat{Y}]\big] + \frac{1}{3!}\Big[\hat{X},\big[\hat{X},[\hat{X},\hat{Y}]\big]\Big] + ... \tag{67}$$

$$e^{i\hat{H}t}\hat{r}e^{-i\hat{H}t} = \hat{r} + [i\hat{H}t,\hat{r}] + \frac{1}{2!}\big[i\hat{H}t,[i\hat{H}t,\hat{r}]\big] + \frac{1}{3!}\Big[i\hat{H}t,\big[i\hat{H}t,[i\hat{H}t,\hat{r}]\big]\Big] + ... \tag{68}$$

Consider $[i\hat{H}t,\hat{r}] = it[\hat{H},\hat{r}]$ one component at a time.

$$\begin{aligned}
[\hat{H},\hat{r}_l] &= \left[\frac{1}{2}\sum_{jk}\hat{r}_j H_{jk}\hat{r}_k,\hat{r}_l\right] && (69)\\
&= \frac{1}{2}\sum_{jk} H_{jk}\big(\hat{r}_j\hat{r}_k\hat{r}_l - \hat{r}_l\hat{r}_j\hat{r}_k\big) && (70)\\
&= \frac{1}{2}\sum_{jk} H_{jk}\big(\hat{r}_j\hat{r}_k\hat{r}_l - \hat{r}_j\hat{r}_l\hat{r}_k + \hat{r}_j\hat{r}_l\hat{r}_k - \hat{r}_l\hat{r}_j\hat{r}_k\big) && (71)\\
&= \frac{1}{2}\sum_{jk} H_{jk}\big(\hat{r}_j[\hat{r}_k,\hat{r}_l] + [\hat{r}_j,\hat{r}_l]\hat{r}_k\big) && (72)\\
&= \frac{1}{2}\sum_{jk} H_{jk}\big(\hat{r}_j i\Omega_{kl} + i\Omega_{jl}\hat{r}_k\big) && (73)\\
&= \frac{i}{2}\sum_{jk} H_{jk}\hat{r}_j\Omega_{kl} + H_{jk}\Omega_{jl}\hat{r}_k && (74)\\
&= \frac{i}{2}\sum_{jk}(-\Omega_{lk})H_{kj}\hat{r}_j + (-\Omega_{lj})H_{jk}\hat{r}_k && (75)\\
&= -i[\Omega H\hat{r}]_l && (76)
\end{aligned}$$

In the above, the first two equalities come from the form of $\hat{H}$. The third equality comes from adding and subtracting the term $\hat{r}_j\hat{r}_l\hat{r}_k$. The fourth equality comes from algebraic simplification, and the fifth equality comes from recognizing that $[\hat{r}_k,\hat{r}_l] = i\Omega_{kl}$. Using the fact that $\Omega$ is antisymmetric, we arrive at the final set of equalities.

This implies

$$\begin{aligned}
[i\hat{H}t,\hat{r}] &= it[\hat{H},\hat{r}] && (77)\\
&= it(-i\Omega H\hat{r}) && (78)\\
&= \Omega H t\hat{r}. && (79)
\end{aligned}$$

Using linearity of the commutator,

$$\begin{aligned}\big[i\hat{H}t,[i\hat{H}t,\hat{r}]\big] &= \big[i\hat{H}t,\Omega Ht\hat{r}\big] && (80)\\ &= (\Omega Ht)^2\hat{r} && (81)\end{aligned}$$

and inductively

$$\Big[i\hat{H}t,...\big[i\hat{H}t,\hat{r}\big]\Big] = (\Omega Ht)^k\hat{r}. \tag{82}$$

In the above, there are $k-1$ nested commutators, or $k$ commutators in total.

This altogether implies

$$\begin{aligned}e^{i\hat{H}t}\hat{r}e^{-i\hat{H}t} &= \sum_{k=0}^{\infty}\frac{(\Omega Ht)^k}{k!}\hat{r} && (83)\\ &= e^{\Omega Ht}\hat{r}, && (84)\end{aligned}$$

concluding the proof. □

**Corollary 10.** *A quadratic Hamiltonian with Hamiltonian matrix $H$ changes the mean vector of a state $\rho$ from $\mathrm{Tr}\big[\hat{r}\rho\big]$ to $e^{\Omega Ht}\mathrm{Tr}\big[\hat{r}\rho\big]$.*

*Proof.* Direct consequence of the above and the following:

$$\begin{aligned}\overline{r}' &= \mathrm{Tr}\Big[\hat{r}e^{-i\hat{H}t}\rho e^{i\hat{H}t}\Big] && (85)\\ &= \mathrm{Tr}\Big[e^{i\hat{H}t}\hat{r}e^{-i\hat{H}t}\rho\Big]. && (86)\end{aligned}$$

Similar to the procedure with the displacement operator, we have reduced the problem of computing the new mean vector to computing $e^{i\hat{H}t}\hat{r}e^{-i\hat{H}t}$, which was accomplished previously.

Finally we can write the effect on the original mean vector:

$$\begin{aligned}\overline{r}' &= \mathrm{Tr}\big[e^{i\hat{H}t}\hat{r}e^{-i\hat{H}t}\rho\big] && (87)\\ &= \mathrm{Tr}\big[e^{\Omega Ht}\hat{r}\rho\big] && (88)\\ &= e^{\Omega Ht}\mathrm{Tr}\big[\hat{r}\rho\big], && (89)\end{aligned}$$

concluding the proof. □

### 3.2 Effect of quadratic Hamiltonian on canonical commutation relations

Now that we have seen the effect of a quadratic Hamiltonian on the mean vector of a state, another natural question to ask is the effect of a quadratic Hamiltonian on the canonical commutation relations. In the following, we redefine $Ht$ as $H$.

**Theorem 11.** *A quadratic Hamiltonian as defined in* (64) *leaves the canonical commutation relations unchanged, i.e.*

$$\Big[e^{\Omega H}\hat{r},\big(e^{\Omega H}\hat{r}\big)^T\Big] = i\Omega. \tag{90}$$

*Proof.* For real symmetric $H$, we have

$$\left[e^{\Omega H}\hat{r}, \left(e^{\Omega H}\hat{r}\right)^T\right] = e^{\Omega H}[\hat{r}, \hat{r}^T](e^{\Omega H})^T \tag{91}$$
$$= e^{\Omega H} i\Omega (e^{\Omega H})^T \tag{92}$$
$$= i\Omega. \tag{93}$$

The first equality follows from the linearity of the commutator. The second equality follows by application of the canonical commutation relation. To see the validity of the last equality, consider the following:

$$e^{\Omega H} i\Omega (e^{\Omega H})^T = e^{\Omega H} i\Omega e^{(\Omega H)^T} \tag{94}$$
$$= e^{\Omega H} i\Omega e^{-H\Omega} \tag{95}$$
$$= i\Omega i\Omega e^{\Omega H} i\Omega e^{-H\Omega} \tag{96}$$
$$= i\Omega e^{(i\Omega)\Omega H (i\Omega)} e^{-H\Omega} \tag{97}$$
$$= i\Omega e^{(i\Omega)(i\Omega) H\Omega} e^{-H\Omega} \tag{98}$$
$$= i\Omega e^{H\Omega} e^{-H\Omega} \tag{99}$$
$$= i\Omega. \tag{100}$$

In the above, the first equality can be seen from the functional calculus of matrices. The second equality comes from the antisymmetry of $\Omega$ and the symmetry of $H$. The third equality arises because $(i\Omega)^2 = I$. The fourth equality also follows from the functional calculus of matrices (i.e. $Mf(X)M^{-1} = f(MXM^{-1})$). The fifth equality comes from combining terms and again recognizing that $(i\Omega)^2 = I$. The sixth and last equalities come from algebraic simplification. □

**Definition 12.** *Any real matrix $S$ for which $S\Omega S^T = \Omega$ is called symplectic; i.e., the action of $S$ preserves the symplectic form $\Omega$.*

**Corollary 13.** *The evolution matrix $e^{\Omega H}$ is symplectic.*

This is seen from the fact that the evolution $e^{\Omega H}$ preserves the canonical commutation relations.

**Lemma 14.** *All symplectic matrices are invertible and the inverse of symplectic $S$ is given by $S^{-1} = -\Omega S^T \Omega$.*

*Proof.* Consider that

$$S\Omega S^T = \Omega \tag{101}$$
$$\Longrightarrow S\Omega S^T \Omega^T = \Omega\Omega^T = I \tag{102}$$
$$\Longrightarrow S^{-1} = \Omega S^T \Omega^T = -\Omega S^T \Omega, \tag{103}$$

concluding the proof. □

### 3.3 Necessity for realness of Hamiltonian matrix

In the preceding defintion of the standard quadratic Hamiltonian as in (64), we restricted ourselves to only real and symmetric $2n \times 2n$ matrices $H$. In the following, we will prove that this is indeed the most general consideration and that antisymmetric Hamiltonian matrices $H$ result in a non-Hermitian Hamiltonian.

**Lemma 15.** *When defining quadratic Hamiltonians, it suffices to restrict ourselves to consider real, symmetric $2n \times 2n$ Hamiltonian matrices.*

*Proof.* To see this, we consider a general Hamiltonian matrix with symmetric and antisymmetric parts, and arrive at the conclusion that the resulting Hamiltonian is not necessarily Hermitian. This will allow us to conclude that to ensure the Hermiticity of the Hamiltonian operator, the corresponding Hamiltonian matrix must be real and symmetric.

Let $H$ be an arbitrary $2n \times 2n$ matrix. We can write it as

$$H = \frac{H + H^T}{2} + \frac{H - H^T}{2} \tag{104}$$
$$= H^s + H^a \tag{105}$$

where $H^s$ denotes the symmetric part of $H$, and $H^a$ denotes the antisymmetric part of $H$. Now consider the operator

$$\frac{1}{2}\hat{r}^T H \hat{r} = \frac{1}{2}\hat{r}^T (H^s + H^a)\hat{r} \tag{106}$$
$$= \frac{1}{2}\hat{r}^T H^s \hat{r} + \frac{1}{2}\hat{r}^T H^a \hat{r}. \tag{107}$$

Focus on the second term in the above.

$$\frac{1}{2}\hat{r}^T H^a \hat{r} = \frac{1}{2}\sum_{jk} \hat{r}_j H^a_{jk} \hat{r}_k \tag{108}$$
$$= \frac{1}{2}\sum_{j<k} \hat{r}_j H^a_{jk} \hat{r}_k + \hat{r}_k H^a_{kj} \hat{r}_j \tag{109}$$
$$= \frac{1}{2}\sum_{j<k} \hat{r}_j H^a_{jk} \hat{r}_k = \hat{r}_k H^a_{jk} \hat{r}_j \tag{110}$$
$$= \frac{1}{2}\sum_{j<k} H^a_{jk} (\hat{r}_j \hat{r}_k - \hat{r}_k \hat{r}_j) \tag{111}$$
$$= \frac{1}{2}\sum_{j<k} H^a_{jk} [\hat{r}_j, \hat{r}_k] \tag{112}$$
$$= \frac{1}{2}\sum_{j<k} H^a_{jk} i\Omega_{jk} \tag{113}$$
$$= \frac{i}{2}\sum_{j<k} H^a_{jk} \Omega_{jk} \tag{114}$$
$$= ic \tag{115}$$

where $c$ is some real number. The key point is that the above term is imaginary.

Thus we see from (107) that if $H^a \neq 0$, then

$$\frac{1}{2}\hat{r}^T H \hat{r} = \frac{1}{2}\hat{r}^T H^s \hat{r} + ic \tag{116}$$

which implies that (107) cannot be Hermitian. We have thus arrived at the desired conclusion. □

### 3.4 Obtaining Hamiltonian matrix from symplectic evolution matrix

If $S = e^{\Omega H}$, then $S$ is symplectic for real, symmetric $H$.

Here, we show a complementary result.

**Theorem 16.** *If $S$ is diagonalizable with strictly positive eigenvalues, then $H = \Omega^T \ln S$ is symmetric, where* ln *denotes the matrix logarithm.*

*Proof.* Consider that

$$\begin{align}
H^T &= \left(\Omega^T \ln S\right)^T \tag{117}\\
&= (\ln S)^T \Omega \tag{118}\\
&= \Omega\Omega^T (\ln S^T)\Omega \tag{119}\\
&= \Omega \ln\left(\Omega^T S^T \Omega\right) \tag{120}\\
&= \Omega \ln\left((-\Omega^T) S^T (-\Omega)\right) \tag{121}\\
&= \Omega \ln\left(\Omega S^T \Omega^T\right) \tag{122}\\
&= \Omega \ln S^{-1} \tag{123}\\
&= -\Omega \ln S \tag{124}\\
&= \Omega^T \ln S \tag{125}\\
&= H. \tag{126}
\end{align}$$

Thus we see that $H = H^T$ and that $H$ is real and symmetric.

The first equality comes from the assumption in the theorem. The second equality comes from distributing the transpose operation. The third equality is apparent when one realizes that $\Omega\Omega^T = I$. The fourth equality comes from the functional calculus of matrices. The fifth equality arises from the antisymmetry of $\Omega$. The seventh equality comes from the fact that $S$ is symplectic. □

**Corollary 17.** *From any symplectic matrix $S$ that is diagonalizable with strictly positive eigenvalues, we can get its Hamiltonian matrix by using $H = \Omega^T \ln S$.*