PHYS 7895: Gaussian Quantum Information

Lecture 6

Lecturer: Mark M. Wilde *Scribe: Eneet Kaur*

1 Overview

In the last lecture, we developed the background required to study single-mode bosonic systems. We studied creation, annihilation, position, and momentum operators and their properties. We then extended the above for multiple-mode bosonic systems, and introduced the canonical symplectic form.

In this lecture, we will introduce the mean vector in Section 2.1 and the covariance matrix of a bosonic state in Section 2.2. We will then derive constraints that are fulfilled by a covariance matrix of a bosonic state in Section 3.

2 Mean vector and covariance matrix

Consider the vector $\hat{r}$ of canonical quadrature operators for an m-mode bosonic system:

$$\hat{r} \equiv (\hat{x}_1, \hat{p}_1 \ldots, \hat{x}_m, \hat{p}_m)^T, \tag{1}$$

where $\hat{x}$ refers to the position-quadrature operator and $\hat{p}$ refers to the momentum-quadrature operator.

2.1 Mean vector

For a state ρ of multiple modes, the mean vector $\overline{r}$ is given by

$$\overline{r} = (\overline{x}_1, \overline{p}_1, \ldots, \overline{x}_m, \overline{p}_m), \tag{2}$$

where the components of the mean vector are defined as follows:

$$\overline{x}_1 = \operatorname{Tr}[\hat{x}_1 \rho] = \operatorname{Tr}\left[\left(\hat{x}_1 \otimes \hat{I} \otimes \ldots \hat{I}\right)\rho\right], \tag{3}$$

$$\overline{p}_1 = \operatorname{Tr}[\hat{p}_1 \rho] = \operatorname{Tr}\left[\left(\hat{I} \otimes \hat{p}_1 \otimes \hat{I} \otimes \ldots \hat{I}\right)\rho\right], \tag{4}$$

$$\overline{x}_j = \operatorname{Tr}[\hat{x}_j \rho] = \langle \hat{x}_j \rangle_\rho, \tag{5}$$

$$\overline{p}_j = \operatorname{Tr}[\hat{p}_j \rho] = \langle \hat{p}_j \rangle_\rho, \tag{6}$$

where $\hat{I}$ is the identity operator and $j \in \{1, 2, \ldots m\}$. Then, as a shorthand we can write the mean vector as

$$\overline{r} = \operatorname{Tr}[\hat{r}\rho] = (\operatorname{Tr}[\hat{x}_1\rho],\ \operatorname{Tr}[\hat{p}_1\rho], \ldots, \operatorname{Tr}[\hat{x}_n\rho],\ \operatorname{Tr}[\hat{p}_n\rho])^T \tag{7}$$

Just like classical probability distributions need not have a finite mean, a quantum state need not have a finite mean.

2.2 Covariance matrix

Let us denote the covariance matrix of a quantum state by σ, and let the entries be given by σ_{jk}. Let $\hat{r}_j$ be the jth element of $\hat{r}$, where $j \in \{1, \ldots 2m\}$, and m is the number of modes of the quantum state considered. Let us define $\hat{r}_j^c = \hat{r}_j - \langle \hat{r}_j \rangle_\rho$. Then, the covariance matrix elements are defined as

$$\sigma_{jk} = \operatorname{Tr}\left[\left(\hat{r}_j^c \hat{r}_k^c + \hat{r}_k^c \hat{r}_j^c\right) \rho\right] \tag{8}$$

$$= \operatorname{Tr}\left[\left\{\hat{r}_j^c, \hat{r}_k^c\right\} \rho\right] \tag{9}$$

$$= \langle \left\{\hat{r}_j^c, \hat{r}_k^c\right\} \rangle_\rho, \tag{10}$$

where $\sigma_{jk} \in \mathbb{R}$ and $k \in \{1, \ldots, 2m\}$.

Now, consider the total photon number operator

$$\hat{N} = \sum_{j=1}^{m} \hat{n}_j, \tag{11}$$

where $\hat{n}_j = \hat{a}_j^\dagger \hat{a}_j$. Let us define finite-energy state as the states that fulfill the following constraint:

$$\operatorname{Tr}\left[\hat{N}\rho\right] < \infty. \tag{12}$$

Proposition 1. *A state has finite energy iff the elements of* $\overline{r}$ *and* σ *are finite, that is* $\overline{r}_j < \infty$ *and* $\sigma_{jk} < \infty$*.*

Proof. Let us first prove that if the state ρ has finite energy then the elements of its mean vector $\overline{r}$ and covariance matrix σ are finite. The definition of a finite-energy state implies

$$\operatorname{Tr}\left[\hat{n}_j \rho\right] < \infty. \tag{13}$$

Then observe that,

$$\operatorname{Tr}\left[\hat{n}_j \rho\right] = \frac{1}{2} \operatorname{Tr}\left[\left(\hat{x}_j^2 + \hat{p}_j^2 - 1\right) \rho\right] < \infty. \tag{14}$$

This implies, $\operatorname{Tr}\left[\hat{x}_j^2 \rho\right], \operatorname{Tr}\left[\hat{p}_j^2 \rho\right] < \infty$. Then, we conclude the following:

$$|\overline{x}_j| = |\operatorname{Tr}\left[\hat{x}_j \rho\right]| \tag{15}$$

$$= |\operatorname{Tr}\left[\hat{x}\sqrt{\rho}\sqrt{\rho}\right]| \tag{16}$$

$$\leq \sqrt{\operatorname{Tr}\left[\hat{x}_j \sqrt{\rho}\sqrt{\rho}\hat{x}_j\right] \cdot \operatorname{Tr}\left[\sqrt{\rho}\sqrt{\rho}\right]} \tag{17}$$

$$= \sqrt{\operatorname{Tr}\left[\hat{x}_j^2 \rho\right]} < \infty. \tag{18}$$

The first inequality follows from the Cauchy–Schwarz inequality. Similarly, $|\overline{p}_j| = |\operatorname{Tr}\left[\hat{p}_j \rho\right]| < \infty$. Therefore, we can conclude that finite-energy states have finite mean vector.

Now, let us prove that the elements of a covariance matrix of finite-energy states are finite. First let us consider the diagonal terms:

$$\begin{aligned}
\sigma_{jj} &= 2\,\mathrm{Tr}\left[\left(\hat{r}_j^c\right)^2\rho\right] && (19)\\
&= 2\,\mathrm{Tr}\left[\left(\hat{r}_j - \langle\hat{r}_j\rangle\right)^2\rho\right] && (20)\\
&= 2\,\mathrm{Tr}\left[\hat{r}_j^2\rho + \langle\hat{r}_j\rangle^2\rho - 2\hat{r}_j\langle\hat{r}_j\rangle\rho\right] && (21)\\
&= 2\,\mathrm{Tr}\left[\hat{r}_j^2\rho - \langle\hat{r}_j\rangle^2\rho\right] && (22)\\
&= 2\left[\langle\hat{r}_j^2\rangle_\rho - \langle\hat{r}_j\rangle_\rho^2\right] && (23)\\
&< \infty. && (24)
\end{aligned}$$

Now, the first term of (23) is finite as seen previously, and the second term is finite since the mean vector of the finite-energy state is finite. Therefore, we conclude that the diagonal elements of a covariance vector of a finite-energy state are finite. Now, we consider the off-diagonal elements σ_{jk}, where $j \neq k$.

$$\begin{aligned}
|\sigma_{jk}| &= |\langle\hat{r}_j^c\hat{r}_k^c + \hat{r}_k^c\hat{r}_j^c\rangle_\rho| && (25)\\
&\leq |\langle\hat{r}_j^c\hat{r}_k^c\rangle_\rho| + |\langle\hat{r}_k^c\hat{r}_j^c\rangle_\rho| && (26)
\end{aligned}$$

Now, consider

$$\begin{aligned}
|\langle\hat{r}_j^c\hat{r}_k^c\rangle_\rho| &= |\,\mathrm{Tr}\left[\sqrt{\rho}\,\hat{r}_j^c\,\hat{r}_k^c\,\sqrt{\rho}\right]| && (27)\\
&\leq \sqrt{\mathrm{Tr}\left[\left(\hat{r}_j^c\right)^2\rho\right]\;\mathrm{Tr}\left[\left(\hat{r}_k^c\right)^2\rho\right]} && (28)\\
&< \infty && (29)
\end{aligned}$$

The first inequality follows from the Cauchy–Schwarz inequality, and the second inequality follows from (19). Now, let us prove the converse. That is, if the state is a finite-energy state, then the covariance matrix is finite.

To prove the opposite implication, consider that

$$\begin{aligned}
\mathrm{Tr}\left(\hat{N}\rho\right) &= \sum_{j=1}^{m}\mathrm{Tr}\left[\hat{n}_j\rho\right] && (30)\\
&= \sum_{j=1}^{m}\left[\mathrm{Tr}\left[\hat{x}_j^2\rho\right] + \mathrm{Tr}\left[\hat{p}_j^2\rho\right] - 1\right] && (31)\\
&< \infty && (32)
\end{aligned}$$

The last inequality follows from the assumed finiteness of the elements of the mena vector and covariance matrix. □

Instead of writing all the $2m \times 2m$ elements of the covariance matrix, we condense it to write the covariance matrix as follows:

$$\sigma = \mathrm{Tr}\left[\left\{\left(\hat{r} - \overline{r}\right), \left(\hat{r} - \overline{r}\right)^\dagger\right\}\rho\right], \tag{33}$$

where,

$$\left\{(\hat{r}-\overline{r}),(\hat{r}-\overline{r})^{\dagger}\right\} = \begin{bmatrix} \{\hat{r}_1-\overline{r}_1,\hat{r}_1-\overline{r}_1\} & \{\hat{r}_1-\overline{r}_1,\hat{r}_2-\overline{r}_2\} & \dots \\ \{\hat{r}_2-\overline{r}_2,\hat{r}_1-\overline{r}_1\} & \{\hat{r}_2-\overline{r}_2,\hat{r}_2-\overline{r}_2\} & \dots \\ \vdots & \vdots & \ddots \end{bmatrix}. \tag{34}$$

Then,

$$\sigma = \begin{bmatrix} \mathrm{Tr}\left[\{\hat{r}_1-\overline{r}_1,\hat{r}_1-\overline{r}_1\}\rho\right] & \mathrm{Tr}\left[\{\hat{r}_1-\overline{r}_1,\hat{r}_2-\overline{r}_2\}\rho\right] & \dots \\ \mathrm{Tr}\left[\{\hat{r}_1-\overline{r}_1,\hat{r}_2-\overline{r}_2\}\rho\right] & \mathrm{Tr}\left[\{\hat{r}_2-\overline{r}_2,\hat{r}_2-\overline{r}_2\}\rho\right] & \dots \\ \vdots & \vdots & \ddots \end{bmatrix}. \tag{35}$$

3 Constraints on covariance matrix

In this section, we establish certain properties of the covariance matrix. We first prove that the covariance matrix (CM) of a vector of random variables is Hermitian and positive semi-definite (PSD). Next, we prove that the covariance matrix of a quantum state fulfills a stronger constraint, that is $\sigma + i\Omega \geq 0$, and that the covariance matrix is positive definite.

3.1 CM of vector of random variables is PSD

Consider a covariance matrix Σ for a vector of random variables. We now prove that the covariance matrix is positive semi-definite.

Proposition 2. *The covariance matrix of a vector of random variables is Hermitian and PSD, that is, $\Sigma = \Sigma^{\dagger}$ and $\Sigma \geq 0$.*

Proof. That the covariance matrix is Hermitian follows from the definition. We now give a proof that the covariance matrix is PSD. Let X be a vector of random variables. Then, $X = [X_1, X_2, \dots, X_m]^T$, where X_i is a random variable and has realizations in $\mathbb{C}$. Then,

$$\Sigma = \mathbb{E}\left[(X-\mathbb{E}(X))\ (X-\mathbb{E}(X))^{\dagger}\right]. \tag{36}$$

Now, let $\underline{w}$ be a constant vector in $\mathbb{C}^m$. Consider then

$$\underline{w}^{\dagger}\Sigma\underline{w} = \underline{w}^{\dagger}\mathbb{E}\left[(X-\mathbb{E}(X))\ (X-\mathbb{E}(X))^{\dagger}\right]\underline{w} \tag{37}$$

$$= \mathbb{E}\left[\underline{w}^{\dagger}(X-\mathbb{E}(X))\ (X-\mathbb{E}(X))^{\dagger}\underline{w}\right] \tag{38}$$

$$= \mathbb{E}\left[|w^{\dagger}(X-\mathbb{E}(X))|^2\right] \geq 0. \tag{39}$$

Since this holds for all $w \in \mathbb{C}^m$, it follows that $\Sigma \geq 0$. □

3.2 Uncertainity principle of covariance matrix

Now, we derive an important constraint on the covariance matrix of a quantum state. This is the **uncertainity principle** for the covariance matrix.

Theorem 3. *The covariance matrix σ of a quantum state fulfills the following constraint:*

$$\sigma + i\Omega \geq 0. \tag{40}$$

Proof. Consider the following $(2m \times 2m)$ complex matrix given by

$$\tau = 2\,\mathrm{Tr}\left[(\hat{r} - \overline{r})\ (\hat{r} - \overline{r})^{\dagger}\,\rho\right]. \tag{41}$$

We first prove that τ is PSD, and then deduce the statement of the theorem. Let $\underline{w} \in \mathbb{C}^{2m}$. Then,

$$\underline{w}^{\dagger}\tau\underline{w} = 2\,\mathrm{Tr}\,\underline{w}^{\dagger}\left[(\hat{r} - \overline{r})\ (\hat{r} - \overline{r})^{\dagger}\,\rho\right]\underline{w} \tag{42}$$

$$= 2\,\mathrm{Tr}\left[\underline{w}^{\dagger}\,(\hat{r} - \overline{r})\ (\hat{r} - \overline{r})^{\dagger}\,\underline{w}\,\rho\right] \tag{43}$$

$$= 2\,\mathrm{Tr}\left[\hat{O}\hat{O}^{\dagger}\rho\right] \tag{44}$$

$$\geq 0, \tag{45}$$

where $\hat{O} = \underline{w}^{\dagger}\,(\hat{r} - \overline{r})$. Since $\hat{O}^{\dagger}\hat{O}$ is PSD and so is ρ, we arrive at the last inequality. Now, the above argument holds for all $\underline{w} \in \mathbb{C}^{2m}$, and so we conclude that τ is PSD.

Now, consider that

$$2\hat{r}_j\hat{r}_k = \{\hat{r}_j, \hat{r}_k\} + [\hat{r}_j, \hat{r}_k]\,. \tag{46}$$

This implies,

$$2\,(\hat{r} - \overline{r})\,(\hat{r} - \overline{r})^{\dagger} = \left\{(\hat{r} - \overline{r})\,, (\hat{r} - \overline{r})^{\dagger}\right\} + \left[(\hat{r} - \overline{r})\,, (\hat{r} - \overline{r})^{\dagger}\right] \tag{47}$$

$$= \left\{(\hat{r} - \overline{r})\,, (\hat{r} - \overline{r})^{\dagger}\right\} + \left[\hat{r}, \hat{r}^{\dagger}\right] \tag{48}$$

Then, we obtain the following:

$$\tau = 2\,\mathrm{Tr}\left[\left((\hat{r} - \overline{r})\,(\hat{r} - \overline{r})^{\dagger}\right)\rho\right] \tag{49}$$

$$= \mathrm{Tr}\left[\left\{(\hat{r} - \overline{r})\,, (\hat{r} - \overline{r})^{\dagger}\right\}\rho\right] + \mathrm{Tr}\left[\left[\hat{r}, \hat{r}^{\dagger}\right]\rho\right] \tag{50}$$

$$= \sigma + i\Omega \geq 0. \tag{51}$$

The last inequality follows from $\tau \geq 0$. □

Now, we prove that σ is PSD. Note that the eigenvalues of a matrix do not change under a transpose. So, if they are positive, then they remain positive after the transpose of the matrix. Then,

$$\sigma + i\Omega \geq 0, \tag{52}$$

implies

$$(\sigma + i\Omega)^{T} \geq 0. \tag{53}$$

which in turn implies,

$$\sigma - i\Omega \geq 0. \tag{54}$$

Then combining (54) with (52), we obtain that $\sigma \geq 0$. That is, σ is a PSD.

3.3 CM of quantum states is positive definite

We now prove that a quantum covariance matrix is in fact positive definite. This makes them more special and easier to work mathematically than classical covariance matrices.

Proposition 4. *A quantum covariance matrix is positive definite.*

Proof. We prove this statement by contradiction. Let us assume that the quantum covariance matrix is not positive definite. That is, $\exists$ a real, non-zero vector $\psi \in \mathbb{R}^{2m}$, such that $\sigma |\psi\rangle = 0$. Then, for $\varepsilon \in \mathbb{R}$, set $\psi(\varepsilon) = (I + \varepsilon i\Omega)\,\psi$. By invoking the following assumption $\sigma\psi = 0$, and the following facts: $\psi^T \Omega \psi = 0\, \forall \psi \in \mathbb{R}^{2m}$ and $(i\Omega)^2 = I$, we find that

$$
\begin{aligned}
&\psi(\varepsilon)^\dagger \left(\sigma + i\Omega\right) \psi(\varepsilon) \\
&= \psi^T \left(I + \varepsilon i\Omega\right)\left(\sigma + i\Omega\right)\left(I + \varepsilon i\Omega\right)\psi && (55)\\
&= \psi^T \left(I + \varepsilon i\Omega\right)\left(i\Omega + \varepsilon\sigma i\Omega + \varepsilon I\right)\psi && (56)\\
&= \psi^T \left(i\Omega + \varepsilon\sigma i\Omega + \varepsilon I + \varepsilon i\Omega\left(i\Omega + \varepsilon\sigma i\Omega + \varepsilon I\right)\right)\psi && (57)\\
&= \psi^T \left(i\Omega + \varepsilon\sigma i\Omega + 2\varepsilon I + \varepsilon^2\Omega^T\sigma\Omega + \varepsilon^2 i\Omega\right)\psi && (58)\\
&= \psi^T \left(2\varepsilon I + \varepsilon^2\Omega^T\sigma\Omega\right)\psi && (59)\\
&= 2\varepsilon\psi^T\psi + \varepsilon^2\left(\Omega\psi\right)^T\sigma\left(\Omega\psi\right) && (60)
\end{aligned}
$$

Now, suppose that $\left(\Omega\psi\right)^T\sigma\left(\Omega\psi\right) = 0$. Then picking $\varepsilon < 0$, implies that $2\varepsilon\psi^T\psi < 0$, which contradicts the fact that $\sigma + i\Omega \geq 0$ for any quantum covariance matrix σ.

Now, suppose that $\left(\Omega\psi\right)^T\sigma\left(\Omega\psi\right) > 0$. Then pick $\varepsilon < 0$ and such that

$$
|\varepsilon| \leq \frac{2\psi^T\psi}{\left(\Omega\psi\right)^T\sigma\left(\Omega\psi\right)}. \tag{61}
$$

This implies,

$$
2\varepsilon\psi^T\psi + \varepsilon^2\left(\Omega\psi\right)^T\sigma\left(\Omega\psi\right) < 0, \tag{62}
$$

and there exists $\psi(\varepsilon)$ such that

$$
\psi(\varepsilon)^\dagger\left(\sigma + i\Omega\right)\psi(\varepsilon) < 0, \tag{63}
$$

again contradicting the assumption that $\sigma + i\Omega \geq 0$. Hence, σ must be positive definite. □

3.4 Uncertainity principle for a single-mode bosonic state

The covariance matrix of a single-mode bosonic state is given as

$$
\sigma = \begin{bmatrix} 2\langle(\hat{x}^c)^2\rangle_\rho & \langle\{\hat{x}^c, \hat{p}^c\}\rangle_\rho \\ \langle\{\hat{x}^c, \hat{p}^c\}\rangle_\rho & 2\langle(\hat{p}^c)^2\rangle_\rho \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}. \tag{64}
$$

The 2×2 matrix σ is the covariance matrix of a single-mode bosonic system if and only if the following constraint holds

$$
\sigma + i\Omega \geq 0 \iff \det(\sigma) \geq 1 \text{ and } \sigma > 0. \tag{65}
$$

The forward direction of the above statement is easy to prove. We have already shown

$$\sigma + i\Omega \geq 0 \implies \sigma > 0. \tag{66}$$

Now we prove that

$$\sigma + i\Omega \geq 0 \implies \det(\sigma) \geq 1. \tag{67}$$

The constraint $\sigma + i\Omega \geq 0$ implies that

$$\det(\sigma + i\Omega) = \sigma_{11}\sigma_{22} - (\sigma_{12}^2 + 1) \geq 0. \tag{68}$$

We thus see that $\det(\sigma) \geq 1$.