PHYS 7895: Gaussian Quantum Information

Lecture 4

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1 Overview

In the last lecture, we discuss continuous functional calculus, polar decomposition of compact bounded operators, unitary operators, exponential maps, trace-class operators, trace norm, Hilbert–Schmidt operators.

In this lecture, we discuss norm topology, weak operator topology, spectral and singular value decompositions for compact operators, duality of trace-class and bounded operators, effects, partial trace, quantum channels, Stinespring dilations, and operator-norm forms. We point readers to [HZ11, Att] for background on topics covered in this lecture.

2 Different notions of convergence

In this section, we discuss different notions of convergence for a sequence of bounded operators to another bounded operator.

Definition 1 (Convergence with respect to uniform topology)**.** Let $\{T_n\}_n \subset \mathcal{L}(\mathcal{H})$ denote sequence of bounded operators and let $T \in \mathcal{L}(\mathcal{H})$ be a bounded operator. Then the sequence $\{T_n\}_n$ converges to T with respect to the uniform or norm topology if

$$\lim_{n\to\infty} \|T_n - T\| = 0 \ . \tag{1}$$

Definition 2 (Convergence with respect to weak operator topology)**.** Let $\{T_n\}_n \subset \mathcal{L}(\mathcal{H})$ denote sequence of bounded operators and let $T \in \mathcal{L}(\mathcal{H})$ be a bounded operator. Then the sequence $\{T_n\}_n$ converges to T with respect to the weak operator topology if for all $\psi, \phi \in \mathcal{H}$

$$\lim_{n\to\infty} |\langle\phi|T_n\psi\rangle - \langle\phi|T\psi\rangle| = 0 \tag{2}$$

Proposition 3. *If a sequence $\{T_j\}_j \subset \mathcal{L}(\mathcal{H})$ converges to $T \in \mathcal{L}(\mathcal{H})$ in norm topology, then it also converges to T weakly.*

Proof. For all $\psi, \phi \in \mathcal{H}$, we have that

$$|\langle\phi|T_j\psi\rangle - \langle\phi|T\psi\rangle| = |\langle\phi|(T_j - T)|\psi\rangle| \tag{3}$$
$$\leq \|\phi\|\|\psi\|\|T_j - T\|. \tag{4}$$

The equality follows from the linearity of operators. The inequality follows from Cauchy-Schwarz inequality and from the definition of the operator norm.

Therefore, if $\lim_{j\to\infty} \|T_j - T\| = 0$, then it follows that $\lim_{j\to\infty} |\langle\phi|T_j\psi\rangle - \langle\phi|T\psi\rangle| = 0$. ☐

We now show an example of a sequence of operators that converges to another operator weakly but does not converge in norm topology.

Example 4. Let $\{\Pi_j\}_j$ be a sequence of orthogonal projections. Let $\{\phi_j\}_{j=1}^{\infty}$ be an orthonormal basis. Then Π_j is projection onto $\text{span}\{\phi_k : k \in \{1, \dots, j\}\}$. Then consider that

$$|\langle\varphi|\Pi_j\psi\rangle - \langle\varphi|\psi\rangle| = |\langle\varphi|(I - \Pi_j)|\psi\rangle|. \tag{5}$$

We now write $|\psi\rangle$ as $|\psi\rangle = \sum_{j=1}^{\infty} \alpha_j|\phi_j\rangle$. Then $(I - \Pi_j)|\psi\rangle = \sum_{l=j+1}^{\infty} \alpha_l|\phi_l\rangle$, so that

$$|\langle\varphi|(I - \Pi_j)|\psi\rangle| = |\langle\phi| \sum_{l=j+1}^{\infty} \alpha_l|\phi_l\rangle| \tag{6}$$

$$\leq \|\phi\| \sum_{l=j+1}^{\infty} |\alpha_l|^2. \tag{7}$$

Since $\lim_{j\to\infty} \sum_{l=j+1}^{\infty} |\alpha_l|^2 = 0$, it implies that $\{\Pi_j\}_j$ converges to I in weak operator topology.

On the other hand, for a fixed j, $\|I - \Pi_j\| = 1$, by picking some unit vector in the space spanned by $I - \Pi_j$. Therefore,

$$\lim_{j\to\infty} \|I - \Pi_j\| = 1, \tag{8}$$

which implies that $\{\Pi_j\}_j$ does not converge to I in norm topology.

Definition 5 (Equivalence of two bounded operators)**.** For operators $A, B \in \mathcal{L}(\mathcal{H})$, if $A = B$, then it should be understood in the weak sense, i.e., $\langle\phi|A\psi\rangle = \langle\phi|B\psi\rangle, \forall\phi, \psi \in \mathcal{H}$.

3 Duality of bounded operators and trace class operators

Definition 6 (Linear functional)**.** A linear mapping f from a complex vector space V to $\mathbb{C}$ is called a linear functional.

Definition 7 (Dual space of a vector space)**.** Let V denote a normed vector space and let V^* denote the set of all continuous linear functionals. Then V^* is called the dual space of V.

A norm on V^* is defined as

$$\|f\| = \sup_{\|v\|=1} |f(v)|. \tag{9}$$

Theorem 8 (Riesz representation theorem)**.** *Let $f \in \mathcal{H}^*$. Then there exists a unique vector $\phi \in \mathcal{H}$ such that $f(\psi) = \langle\phi|\psi\rangle$. Moreover, $\|f\| = \|\phi\|$.*

We now extend the discussion on the dual space of trace-class operators. Let $S \in \mathcal{L}(\mathcal{H})$ and let $T \in \mathcal{T}(\mathcal{H})$. Then a linear functional f_S on $\mathcal{T}(\mathcal{H})$ can be defined as

$$f_S(T) = \text{Tr}\{ST\} . \tag{10}$$

Theorem 9. *The mapping $S \to f_S$ is a linear bijection from $\mathcal{L}(\mathcal{H})$ to $\mathcal{T}(\mathcal{H})^*$, and $\|S\| = \|f_S\|, \forall S \in \mathcal{L}(\mathcal{H})$.*

Moreover, we can conclude the following:

1. $S \geq 0 \Leftrightarrow f_S(T) \geq 0, \forall T \geq 0$.
2. $S = S^\dagger \Leftrightarrow f_S(T) \in \mathbb{R}, \forall T = T^\dagger$.

4 Quantum Mechanics

4.1 Quantum states

A set $\mathcal{S}(\mathcal{H})$ of quantum states is defined as

$$\mathcal{S}(\mathcal{H}) = \{\rho \in \mathcal{T}(\mathcal{H}) : \rho \geq 0,\ \mathrm{Tr}\{\rho\} = 1\}. \tag{11}$$

Theorem 10. *A quantum state $\rho \in \mathcal{S}(\mathcal{H})$ has a canonical convex decomposition of the form*

$$\rho = \sum_j \lambda_j P_j, \tag{12}$$

where $\{\lambda_j\}_j$ is a finite or an infinite sequence of positive numbers, such that $\sum_j \lambda_j = 1$, and $\{P_j\}_j$ is a set of orthogonal projections.

4.2 Effect

Effect is a mapping from the set of states $\mathcal{S}(\mathcal{H})$ to the interval $[0,1]$, i.e., $\rho \to E(\rho) \in [0,1]$. $E(\rho)$ is the probability of a "yes" answer to "the recorded measurement outcome belongs to a subset $X \subset \Omega$."

Basic assumption behind an effect is the following:

$$E(\lambda\rho_1 + (1-\lambda)\rho_2) = \lambda E(\rho_1) + (1-\lambda)E(\rho_2), \forall \rho_1, \rho_2 \in \mathcal{S}(\mathcal{H}), \lambda \in [0,1], \tag{13}$$

Proposition 11. *Let E be an effect. Then there exists $\hat{E} \in \mathcal{L}_S(\mathcal{H})$ such that $E(\rho) = \mathrm{Tr}[\hat{E}\rho], \forall \rho \in \mathcal{S}(\mathcal{H})$, where $0 \leq \hat{E} \leq I$.*

4.3 Partial trace

Definition 12 (Partial trace)**.** $\mathrm{Tr}_A : \mathcal{T}(\mathcal{H}_A \otimes \mathcal{H}_B) \to \mathcal{T}(\mathcal{H}_B)$ is a linear mapping satisfying

$$\mathrm{Tr}\{\mathrm{Tr}_A\{T_{AB}\}E_B\} = \mathrm{Tr}\{T_{AB}(I_A \otimes E_B)\}, \tag{14}$$

$\forall T_{AB} \in \mathcal{T}(\mathcal{H}_A \otimes \mathcal{H}_B)$ and $E_B \in \mathcal{L}(\mathcal{H}_B)$.

The partial trace can be calculated as follows. Let $\{\psi_j\}_j$ and $\{\phi_k\}_k$ denote orthonormal bases for $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively. Then

$$\mathrm{Tr}_A\{T\} = \sum_{j,k,n} \left[\langle \psi_j|_A \otimes \langle \phi_k|_B T_{AB} |\psi_j\rangle_A \otimes |\phi_n\rangle_B \right] |\phi_k\rangle\langle\phi_n|_B. \tag{15}$$

4.4 State Purification

Let $\rho_A \in \mathcal{S}(\mathcal{H})$ denote a quantum state. Then a purification of ρ_A is a vector $|\psi\rangle_{RA} \in \mathcal{H}_R \otimes \mathcal{H}_A$ such that $\mathrm{Tr}_R\{|\psi\rangle\langle\psi|_{RA}\} = \rho_A$.

A purification of ρ_A can be constructed from the spectral decomposition of ρ_A.

$$\rho = \sum_j \lambda_j |\psi_j\rangle\langle\psi_j|_A \ , \tag{16}$$

where $\{|\psi_j\rangle\}_j$ is an orthonormal basis, as

$$|\psi\rangle_{RA} = \sum_j \sqrt{\lambda_j} |\psi_j\rangle_R |\psi_j\rangle_A. \tag{17}$$

4.5 Quantum channels

Definition 13 (Positivity of a linear map)**.** A linear mapping $\mathcal{N}_{A\to B} : \mathcal{T}(\mathcal{H}_A) \to \mathcal{T}(\mathcal{H}_B)$ is positive if $\mathcal{N}(T) \geq 0, \forall T \geq 0, T \in \mathcal{T}(\mathcal{H})$.

Definition 14 (Complete positivity of a linear map)**.** A linear map $\mathcal{N}_{A\to B} : \mathcal{T}(\mathcal{H}_A) \to \mathcal{T}(\mathcal{H}_B)$ is completely positive if $\mathrm{id}_R \otimes \mathcal{N}_{A\to B}$ is positive for all finite-dimensional $\mathcal{H}_R$.

Definition 15 (Quantum channel)**.** A linear map $\mathcal{N}_{A\to B} : \mathcal{T}(\mathcal{H}_A) \to \mathcal{T}(\mathcal{H}_B)$ is a quantum channel if it is completely positive and trace preserving.

Definition 16 (Adjoint of a linear map)**.** Let $\mathcal{N}_{A\to B} : \mathcal{T}(\mathcal{H}_A) \to \mathcal{T}(\mathcal{H}_B)$ be a linear map. The adjoint $\mathcal{N}^\dagger : \mathcal{L}(\mathcal{H}_B) \to \mathcal{L}(\mathcal{H}_A)$ of a linear map $\mathcal{N}$ is a unique linear map satisfying the following set of equations:

$$\mathrm{Tr}\{\mathcal{N}(T)E\} = \mathrm{Tr}\{T\mathcal{N}^\dagger(E)\}, \tag{18}$$

$\forall T \in \mathcal{T}(\mathcal{H})$ and $E \in \mathcal{L}(\mathcal{H})$.

4.5.1 Stinespring dilation

Definition 17. Let $\mathcal{H}_A$, $\mathcal{H}_B$ and $\mathcal{H}_E$ be Hilbert spaces, and let $\mathcal{N} : \mathcal{T}(\mathcal{H}_A) \to \mathcal{T}(\mathcal{H}_B)$ be a quantum chanenel. An isometric extension or Stinespring dilation $V \in \mathcal{L}(\mathcal{H}_A) \to \mathcal{L}(\mathcal{H}_B \otimes \mathcal{H}_E)$ of the channel $\mathcal{N}$ is a linear isometry such that

$$\mathcal{N}(X_A) = \mathrm{Tr}_E[V X_A V^\dagger], \tag{19}$$

for all $X_A \in \mathcal{T}(\mathcal{H}_A)$.

4.5.2 Operator-sum form

Proposition 18. *A map $\mathcal{N} : \mathcal{T}(\mathcal{H}_A) \to \mathcal{T}(\mathcal{H}_B)$ is a quantum channel if and only if there exists a sequence of bounded operators $\{A_k\}_k$ such that*

$$\mathcal{N}(T) = \sum_k A_k T A_k^\dagger, \tag{20}$$

$\sum_k A_k^\dagger A_k = I, \forall T \in \mathcal{T}(\mathcal{H}_A)$.

References

[Att] Stephane Attal. Lectures in quantum noise theory. `http://math.univ-lyon1.fr/~attal/chapters.html`.

[HZ11] Teiko Heinosaari and Mario Ziman. *The Mathematical Language of Quantum Theory: From Uncertainty to Entanglement*. Cambridge University Press, 2011. `https://www.cambridge.org/core/books/mathematical-language-of-quantum-theory/D8AAEF727B99D7AB098F9162C6D55FC8`.