PHYS 7895: Gaussian Quantum Information

Lecture 3

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1 Overview

In the last lecture, we defined separable Hilbert spaces, bounded operators, operator norms, C*-algebra, the spectrum of bounded operators, and self-adjoint and positive operators.

In this lecture, we discuss continuous functional calculus, polar decomposition of compact bounded operators, unitary operators, exponential maps, trace-class operators, trace norm, Hilbert–Schmidt operators. We point readers to [HZ11, Att] for background on topics covered in this lecture.

2 Properties of bounded linear operators

In this section, we continue our discussion on the properties of bounded linear operators. We begin by showing that in a limiting sense, the square root of any bounded positive semi-definite (PSD) operator can be defined.

2.1 Square root of a bounded positive semi-definite operator

Unlike the case of finite-dimensional Hilbert spaces, a bounded self-adjoint operator acting on an infinite-dimensional separable Hilbert space need not have a spectral decomposition. We will discuss later that the spectral decomposition only holds for a subclass of bounded linear operators that are called *compact* bounded operators. We recall from the previous lecture that a bounded operator may not necessarily have eigenvalues, but it has a spectrum.

Let $\mathcal{H}$ denote a separable Hilbert space, let $\mathcal{L}(\mathcal{H})$ denote the set of bounded operators, and let $\mathcal{L}_S(\mathcal{H})$ denote the set of bounded self-adjoint operators. An operator $T \in \mathcal{L}(\mathcal{H})$ is positive semi-definite (PSD) if $\langle \psi | T\psi \rangle \geq 0, \forall \psi \in \mathcal{H}$. Moreover, PSD operators are also self-adjoint operators.

Lemma 1. *Let $T \in \mathcal{L}_S(\mathcal{H})$ be a bounded PSD operator. Then there is a unique bounded PSD operator $\sqrt{T}$ satisfying $(\sqrt{T})^2 = T$.*

Proof. Instead of giving a complete proof, we just provide a sketch of the proof here. We point readers to [Att] for a detailed review of the proof.

As discussed in the previous lecture, if T is a self-adjoint operator, then the spectrun of T exists. Let $\sigma(T)$ denote the spectrum of T. In particular, if T is a PSD operator, then $\sigma(T) \subset \mathbb{R}_{\geq 0}$.

For an operator A acting on a finite-dimensional Hilbert space, one can find $\sqrt{A}$ by applying the square-root function on eigenvalues. Although the same notion of functional calculus may not hold for bounded operators acting on a separable Hilbert space, we can approximate the square root of an operator by using polynomial functions. For the proof sketch, we recall the key properties of bounded operators. The operator norm of a bounded operator is finite. Moreover, from the homogeneity and triangle inequality of the operator norm, any linear combination of bounded operators is bounded. Furthermore, from the submultiplicativity of the operator norm, the multiplication of two bounded operators is also bounded.

The continuous functional calculus can be applied to any bounded operator. Let p denote a polynomial function. Then a polynomial function of a bounded operator T is defined as $p(T)$. For example, $p(T) = T + 2T^2 + 4T^3$ is a well defined polynomial function of T.

We now state an important theorem concerning the uniform convergence of polynomial functions to an arbitrary continuous function on a bounded interval.

Stone-Weierstrass theorem: Suppose f is a continuous real-valued function defined on the real interval $x \in [a, b]$. Then for every $\varepsilon > 0, \exists N_\varepsilon \in \mathbb{Z}^+$, such that $\forall n > N_\varepsilon$, the following holds

$$|p_n(x) - f(x)| \leq \varepsilon, \forall x \in [a, b], \tag{1}$$

where p_n denotes a polynomial function with degree n.

The basic idea is that there exists a sequence of polynomials $\{p_n\}_n$, such that a continuous function f can be defined as a limit of this sequence.

We now argue that there exists an explicit construction of the sequence of polynomials approximating a continuous function f. Let f denote a continuous function on the interval $x \in [0, 1]$. Consider the following Bernstein polynomial:

$$B_n(f)(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k} \ . \tag{2}$$

By using the law of large numbers and Chebyshev's inequality, it can be shown that $B_n(f)$ converges uniformly to f, i.e.,

$$\lim_{n\to\infty} \sup\{|B_n(f)(x) - f(x)| : x \in [0, 1]\} = 0 \ . \tag{3}$$

As discussed earlier, for a PSD operator T, $\sigma(T) \in [0, \|T\|]$. Moreover, the function $f : x \to \sqrt{x}$ is continuous on $\sigma(T)$. Therefore, $\sqrt{T}$ can be defined as a limit of the sequence of Bernstein polynomials. Moreover, it can be shown that $\sqrt{T}$ is unique. □

2.2 Compactness of bounded operators

Definition 2. A bounded operator $T \in \mathcal{L}(\mathcal{H})$ is compact if for all bounded sequences $\{\psi_n\}_n$, $\{\|T\psi_n\|\}_n$ has a convergent subsequence. Equivalently, $T \in \mathcal{L}(\mathcal{H})$ is compact if for all orthonormal bases $\{\psi_j\}_j$,

$$\lim_{j\to\infty} \|T\psi_j\| = 0. \tag{4}$$

2.3 Absolute value of a bounded operator

In Section 2.1, we showed that every bounded operator has a unique square root. By using the square root function, the absolute value of an operator can be defined.

Definition 3. Let $T \in \mathcal{L}(\mathcal{H})$. The absolute value of the operator T is defined as

$$|T| \equiv \sqrt{T^{\dagger}T}. \tag{5}$$

2.4 Polar decomposition of bounded operators

The absolute value of a bounded operator as defined in Section 2.3 can be used to define the polar decomposition of bounded operators.

Lemma 4. *Let $T \in \mathcal{L}(\mathcal{H})$. Then there exists a bounded operator $V \in \mathcal{L}(\mathcal{H})$ such that $T = V|T|$, where $\|V\psi\| = \|\psi\|, \forall \psi \in \operatorname{supp}(V)$.*

Proof. Consider the following chain of equalities for all $\psi, \phi \in \mathcal{H}$.

$$\langle |T|\psi | |T|\phi\rangle = \langle \psi | |T|^2 \phi\rangle \tag{6}$$

$$= \langle \psi | T^{\dagger}T\phi\rangle \tag{7}$$

$$= \langle T\psi | T\phi\rangle. \tag{8}$$

The first equality follows from the definition of the adjoint of $|T|$ and from the fact that $|T|^{\dagger} = |T|$. The second equality follows from the definition of the absolute value of T as defined in (5). The last equality follows from the definition of the adjoint of $T^{\dagger}$.

Therefore, the mapping is such that all inner products are preserved, i.e., the mapping is an isometry. This completes the proof. □

We note that the isometry V in Lemma 4 is from $\operatorname{ran}(|T|)$ to $\operatorname{ran}(T)$. Moreover, it is a partial isometry in the sense that it is 0 for all the vectors in $\operatorname{ran}^{\perp}(|T|)$.

2.4.1 Polar decomposition of compact bounded operators

In Lemma 4, we showed that for a bounded operator T, there exists an isometry V, such that $T = V|T|$. In this section, we argue that for a compact bounded operator, an explicit form of V can be defined in terms of orthonormal basis vectors.

Let $T \in \mathcal{L}(\mathcal{H})$ be a compact bounded operator. Then

$$T = \sum_{n=0}^{\infty} \lambda_n |\phi_n\rangle\langle\psi_n|, \tag{9}$$

where the sequence $\{\lambda_n\}_n \subset R^{+}\backslash\{0\}$ is either finite or converges to zero, and $\{|\phi_n\rangle\}_n$ and $\{|\psi_n\rangle\}_n$ are orthonormal basis elements.

Then $|T|$ is given by

$$|T| = \sum_{n=0}^{\infty} \lambda_n |\psi_n\rangle\langle\psi_n| \ . \tag{10}$$

Moreover, an isometry V such that $T = V|T|$, is defined as

$$V = \sum_{n=0}^{\infty} |\phi_n\rangle\langle\psi_n| \ . \tag{11}$$

2.5 Unitary operators

We recall the definition of isomorphism from previous lectures. U is an isomorphism if the following holds for all $\phi, \psi \in \mathcal{H}$:

$$\langle U\phi | U\psi \rangle = \langle \phi | \psi \rangle. \tag{12}$$

Moreover, the operator norm of U is defined as

$$\|U\| = \sup_{\psi : \|\psi\| = 1} \|U\psi\| = 1. \tag{13}$$

The first equality follows from the definition of the operator norm. The second equality follows from (12) by setting $\phi = \psi$.

Definition 5 (Unitary operators)**.** A bounded operator $U \in \mathcal{L}(\mathcal{H})$ is unitary if $UU^\dagger = U^\dagger U = I$.

Theorem 6. *Let U be a linear map on $\mathcal{H}$. Then following assertions are equivalent:*

1. *U is an isomorphism.*
2. *U is an onto isometry.*
3. *U is bounded and $UU^\dagger = U^\dagger U = I$.*

Proof. 1. $\Rightarrow$ 2.: It follows from (12) by picking $\phi = \psi$, so that $\|U\psi\| = \|\psi\|, \forall \psi \in \mathcal{H}$. Therefore, U is an onto isometry.

2. $\Rightarrow$ 3.: Since U is an isomtery, then for all $\psi, \phi \in \mathcal{H}$, the following holds

$$\|U\psi - U\phi\| = \|\psi - \phi\| \ . \tag{14}$$

Therefore, $U\phi = U\psi$ if and only if $\psi = \phi$, which implies that U is a bijective map.

Let $\phi \in \mathcal{H}$. Consider the following chain of equalities:

$$\langle \phi | \phi \rangle = \langle U\phi | U\phi \rangle \tag{15}$$
$$= \langle \phi | U^\dagger U \phi \rangle \ . \tag{16}$$

The first equality follows since U is an isometry. The last equality follows from the definition of the adjoint of U. From (16), it follows that $U^\dagger U = I$. Therefore, $U^-1 = U^\dagger$, which further implies that $UU^\dagger = I$.

3. $\Rightarrow$ 1.: Let $U \in \mathcal{L}(\mathcal{H})$, such that $U^\dagger U = UU^\dagger = I$. Then for all $\psi, \phi \in \mathcal{H}$, the following holds:

$$\begin{aligned} \langle \phi | \psi \rangle &= \langle \phi | U^\dagger U \psi \rangle && (17) \\ &= \langle U\phi | U\psi \rangle , && (18) \end{aligned}$$

which implies that U is an isomorphism. □

2.5.1 Eigenvalues of a unitary operator

Suppose that $U\psi = \lambda\psi$ for some non-zero $\psi \in \mathcal{H}$. Then

$$\begin{aligned} \langle \psi | \psi \rangle &= \langle \psi | U^\dagger U \psi \rangle && (19) \\ &= \langle \psi | \lambda^* \lambda \psi \rangle && (20) \\ &= |\lambda|^2 \langle \psi | \psi \rangle \ , && (21) \end{aligned}$$

which implies that $|\lambda| = 1$.

2.6 Connection between unitary and self-adjoint operators

In this section, we define the notion of exponential maps on $\mathcal{L}(\mathcal{H})$. Let $T \in \mathcal{L}(\mathcal{H})$ be a bounded operator. For $k \in \mathbb{N}$, define

$$F_k(T) \equiv \sum_{n=0}^{k} \frac{T^n}{n!}. \qquad (22)$$

We note that $F_k(T)$ is a legitimate bounded operator for a finite k, which follows from the triangle inequality and the sub-multiplicativity of the operator norm.

Consider the following positive-valued function:

$$f_k(T) \equiv \sum_{n=0}^{k} \frac{\|T^n\|}{n!}, \qquad (23)$$

where $T^0 = I$.

Consider the following chain of inequalities:

$$\begin{aligned} f_k(T) &\leq \sum_{n=0}^{k} \frac{\|T\|^n}{n!} && (24) \\ &\leq \sum_{n=0}^{\infty} \frac{\|T\|^n}{n!} && (25) \\ &= e^{\|T\|} && (26) \\ &< \infty \ . && (27) \end{aligned}$$

The first inequality follows from the sub-multiplicativity of the operator norm. The second inequality follows as a sum of positive numbers is greater than 0. The first equality follows from the Taylor series expansion of the function e^x. The last strict inequality follows from the fact that the operator norm of a bounded operator is finite.

Form the aforementioned series of arguments, along with the triangle inequality for the operator norm, it follows that the series $\sum_{n=0}^{\infty} \frac{T^n}{n!}$ is absolutely convergent. Therefore, the exponential map of a bounded operator can be defined as

$$e^T \equiv \lim_{k\to\infty} F_k(T) \ . \tag{28}$$

For $T \in \mathcal{L}(\mathcal{H})$, and for all $a, b \in \mathbb{C}$, the following properties hold

$$e^{aT} e^{bT} = e^{(a+b)T}, \tag{29}$$

$$(e^{aT})^\dagger = e^{\bar{a}T^\dagger}. \tag{30}$$

For $T \in \mathcal{L}_S(\mathcal{H})$, $(e^{iT})^\dagger = e^{-iT}$, and $e^{iT}e^{-iT} = e^0 = I$, which implies that e^{iT} is a unitary operator.

2.7 Normal Operators

Definition 7 (Normal Operators)**.** Let $T \in \mathcal{L}(\mathcal{H})$ be a bounded operator. Then T is normal if

$$TT^\dagger = T^\dagger T. \tag{31}$$

It is easy to check that both self-adjoint and unitary operators are normal operators.

For normal operators that are also compact, there is a spectral decomposition, i.e., there exists a sequence $\{\lambda_j\}_j$ of complex numbers and an orthonormal basis $\{\phi_j\}_j$ such that

$$T = \sum_{j=1}^{\infty} \lambda_j |\phi_j\rangle\langle\phi_j|. \tag{32}$$

Moreover, the action of T on a vector $|\psi\rangle \in \mathcal{H}$ is given by

$$T|\psi\rangle = \sum_{j=1}^{\infty} \lambda_j \langle\phi_j|\psi\rangle|\phi_j\rangle. \tag{33}$$

2.8 Trace-class operators

The trace is meaningful only for a subset of bounded operators.

Definition 8 (Trace of PSD operators)**.** Let $\mathcal{H}$ be a separable Hilbert space and $\{\phi_j\}_{j=1}^{\infty}$ be an orthonormal basis. Then for a PSD operator $T \in \mathcal{L}(\mathcal{H})$,

$$\mathrm{Tr}\{T\} = \sum_{j=1}^{\infty} \langle\phi_j|T\phi_j\rangle. \tag{34}$$

Due to $T \geq 0$, the trace of T is a sum of non-negative numbers. Therefore, if the sum does not converge, $\text{Tr}\{T\} = \infty$.

Theorem 9. *Let $T \in \mathcal{L}(\mathcal{H})$ be a PSD operator. Then* $\text{Tr}\{T\}$ *does not depend on the choice of orthonormal basis.*

Proof. Let $\{\phi_j\}_{j=1}^{\infty}$ and $\{\psi_j\}_{j=1}^{\infty}$ denote two different orthonormal basis. Consider the following chain of equalities:

$$\sum_j \langle \psi_j | T \psi_j \rangle = \sum_j \|T^{1/2}\psi_j\|^2 \tag{35}$$

$$= \sum_j \sum_k |\langle \phi_k | T^{1/2}\psi_j \rangle|^2 \tag{36}$$

$$= \sum_k \sum_j |\langle \psi_j | T^{1/2}\phi_k \rangle|^2 \tag{37}$$

$$= \sum \|T^{1/2}\phi_k\|^2 \tag{38}$$

$$= \sum_k \langle \phi_k | T \phi_k \rangle. \tag{39}$$

The first equality follows from Lemma 1. The second equality follows from Parseval's formula for the norm of a vector. The third equality follows from Tonelli's theorem. The fourth equality follows again from Parseval's formula. □

Definition 10 (Trace-class operators)**.** A bounded operator $T \in \mathcal{L}(\mathcal{H})$ is trace-class if

$$\text{Tr}\{|T|\} < \infty \; . \tag{40}$$

We denote the trace class operators acting on a separable Hilbert space $\mathcal{H}$ by $\mathcal{T}(\mathcal{H})$. We now provide two examples of bounded operators that are not trace-class operators.

Example 11. The identity operator I is bounded but is not a trace-class operator since $\text{Tr}\{|I|\} = \infty$.

Example 12. Let A denote a shift operator. Then $|A| = (A^\dagger A)^{1/2} = I^{1/2} = I$, which implies that

$$\sum_j \langle \delta_j | \sqrt{A^\dagger A}\, \delta_j \rangle = \sum_j \langle \delta_j | I \delta_j \rangle \tag{41}$$

$$= \infty. \tag{42}$$

Therefore, A is not a trace-class operator.

Theorem 13. *Let $T \in \mathcal{T}(\mathcal{H})$ and let $\{\phi_j\}_{j=1}^{\infty}$ be an orthonormal basis. Then* $\text{Tr}\{T\} = \sum_{j=1}^{\infty} \langle \phi_j | T \phi_j \rangle$ *is the trace of the operator T and is independent of the basis chosen.*

Proof. We begin by showing that every trace-class operator is compact. Let T be a positive trace-class operator. Let $\{\phi_j\}_{j=1}^{\infty}$ be an orthonormal basis. Consider the following chain of inequalities:

$$\sum_j \|\sqrt{T}\phi_j\|^2 = \sum_j \langle \phi_j | T \phi_j \rangle \tag{43}$$

$$= \text{Tr}\{T\} \tag{44}$$

$$< \infty \; . \tag{45}$$

The second equality follows from Definition 8. The strict inequality follows because T is a trace-class operator. Therefore, $\|\sqrt{T}\phi_j\| \to 0$ as $j \to \infty$, which implies that $\sqrt{T}$ is compact, which further implies that T is compact.

Now for an arbitrary trace-class operator T, $|T|$ is also trace-class, and since $|T|$ is positive, from the aforementioned arguments it follows that $|T|$ is compact. Moreover, from the polar decomposition, $T = U|T|$, we get that T is compact.

Let $\{\psi_j\}_j$ denote an orthonormal basis. Then, from (9), a trace-class operator $T \in \mathcal{T}(\mathcal{H})$ can be written as

$$T = \sum_j \lambda_j |\psi_j\rangle\langle\phi_j|. \tag{46}$$

Consider the following chain of inequalities:

$$
\begin{aligned}
|\operatorname{Tr}\{T\}| &= |\sum_k \langle\varphi_k|T\varphi_k\rangle| && (47)\\
&\leq \sum_k |\langle\varphi_k|T\varphi_k\rangle| && (48)\\
&= \sum_{k,j} \lambda_j |\langle\varphi_k|\psi_j\rangle||\langle\phi_j|\varphi_k\rangle| && (49)\\
&\leq \sum_j \lambda_j [\sum_k |\langle\varphi_k|\psi_j\rangle|^2]^{1/2} [\sum_k |\langle\varphi_k|\phi_j\rangle|^2]^{1/2} && (50)\\
&= \sum_j \lambda_j \|\psi_j\| \|\phi_j\| && (51)\\
&= \sum_j \lambda_j && (52)\\
&= \operatorname{Tr}[|T|] < \infty. && (53)
\end{aligned}
$$

The first inequality follows as the absolute value a sum is smaller than a sum of absolute values. The second equality follows from (46). The second inequality follows from the Cauchy-Schwarz inequality. The third equality follows from the Parseval's formula.

Therefore, the aforementioned arguments establish the absolute convergence of $\operatorname{Tr}\{T\}$. Then from Fubini's theorem, the following holds

$$
\begin{aligned}
\sum_k \langle\varphi_k|T\varphi_k\rangle &= \sum_k \sum_j \lambda_j \langle\varphi_k|\psi_j\rangle\langle\phi_j|\varphi_k\rangle && (54)\\
&= \sum_j \lambda_j \sum_k \langle\phi_j|\varphi_k\rangle\langle\varphi_k|\psi_j\rangle && (55)\\
&= \sum_j \lambda_j \langle\phi_j|\psi_j\rangle. && (56)
\end{aligned}
$$

Therefore, the trace does not depend on the choice of $\{|\varphi_k\rangle\}_k$.

□

Definition 14 (Trace norm)**.** Let $T \in \mathcal{T}(\mathcal{H})$ be a trace-class operator. Then the trace norm is defined as

$$\|T\|_1 \equiv \operatorname{Tr}\{|T|\} \ . \tag{57}$$

Lemma 15. *Let* $T \in \mathcal{T}(\mathcal{H})$*. Then*

$$\|T\|_1 \equiv \sup_{U \in \mathrm{U}(\mathcal{H})} |\operatorname{Tr}[UT]|, \tag{58}$$

where $\mathrm{U}(\mathcal{H})$ *denotes a set of bounded unitary operators.*

The following relation holds between the operator norm and the trace norm for all $T \in \mathcal{T}(\mathcal{H})$:

$$\|T\| \leq \|T\|_1 \ . \tag{59}$$

Let $S \in \mathcal{L}(\mathcal{H})$ be a bounded operator. Even though $\operatorname{Tr}\{S\}$ is not finite for all bounded operators, $\operatorname{Tr}\{ST\}$ is finite whenever $T \in \mathcal{T}(\mathcal{H})$. It follows from the following inequality:

$$|\operatorname{Tr}\{ST\}| \leq \|T\|_1 \|S\| \ , \tag{60}$$

which follows from Holder's inequality.

Definition 16 (Hilbert-Schmidt norm)**.** Let $T \in \mathcal{L}(\mathcal{H})$. Then Hilbert-Schmidt norm of T is defined as

$$\|T\|_2 = \|T\|_{\mathrm{HS}} \equiv \operatorname{Tr}\{T^\dagger T\}^{1/2} \ . \tag{61}$$

Moreover, Hilbert-Schmidt operators are those for which

$$\|T\|_2 < \infty. \tag{62}$$

The following relation holds between different norms of operators acting on a separable Hilbert space.

$$\|T\| \leq \|T\|_2 \leq \|T\|_1 . \tag{63}$$

Moreover, from Cauchy-Schwarz inequality, we get

$$|\operatorname{Tr}\{ST\}|^2 \leq \|S\|_2^2 \|T\|_2^2 \ . \tag{64}$$

References

[Att] Stephane Attal. Lectures in quantum noise theory. `http://math.univ-lyon1.fr/~attal/chapters.html`.

[HZ11] Teiko Heinosaari and Mario Ziman. *The Mathematical Language of Quantum Theory: From Uncertainty to Entanglement*. Cambridge University Press, 2011. `https://www.cambridge.org/core/books/mathematical-language-of-quantum-theory/D8AAEF727B99D7AB098F9162C6D55FC8`.