PHYS 7895: Gaussian Quantum Information

Lecture 2

Lecturer: Mark M. Wilde *Scribe: Anthony Brady*

1 Overview

The focus of Lecture [reference Lecture 1] was defining and discussing the properties of a separable Hilbert space $\mathcal{H}$, the arena in which physical states of quantum theory reside and evolve (á la the first postulate of quantum mechanics). The focus of this lecture however is not on $\mathcal{H}$ itself; rather we direct our focus towards the entities which *act on* elements of $\mathcal{H}$, i.e. the operators on $\mathcal{H}$. The role that operators play in the quantum theory is ever encompassing and knowledge of their properties is elemental to understanding the foundations of the theory. Also, perhaps less fundamental but more practical, the operator language developed here will be used in later lectures, computations, etc. To solidify/motivate the former point a bit, consider two primary examples: 1) the second postulate of quantum mechanics states that physical observables (e.g. experimental data!) correspond to self-adjoint (or Hermitian) operators on $\mathcal{H}$, and 2) physical states themselves are generically represented by 'density operators' which are *trace class*.[1] And the list goes on. From here it seems obvious to reason that a solid understanding of operators and operator language is worthy to develop. That is the goal of this lecture.

2 Operators on a Hilbert Space

We focus on **bounded linear operators** here, although we will deal quite often with unbounded operators as well. Linearity is forced upon us by the postulates of quantum theory. So all operators in these notes are presumed to be linear. Recall that linearity is defined in the usual manner: a mapping/operator $T : \mathcal{H} \to \mathcal{H}$ is linear iff $T(c\psi + \varphi) = cT(\psi) + T(\varphi)$ $\forall$ $\psi, \varphi \in \mathcal{H}$ and $c \in \mathbb{C}$. We will use the notation $T(\psi) = T\psi$ from here out. A mapping is then said to be **bounded** if $\exists$ $t \geq 0$ such that:

$$\|T\psi\| \leq t\|\psi\| \tag{1}$$

$\forall$ $\psi \in \mathcal{H}$. We let $\mathfrak{L}(\mathcal{H})$ denote the set of all bounded operators.

As mentioned above, all operators dealt with here are linear, however not all are necessarily bounded. A simple counterexample is the number operator $\hat{n}$ of the simple harmonic oscillator. Perhaps it is intuitive that the number operator is unbounded as the ladder of the simple harmonic oscillator has no upper rung. However, we will explicitly show that $\hat{n}$ is unbounded by using relation (1) together with a simple counterexample. We need only construct one state $\beta \in \mathcal{H}$ which violates the inequality (1) to do the job.

[1] The terms self-adjoint and trace class will be defined in detail in this lecture and Lecture 3, respectively.

Example 1. *Consider the* Basal state*:*

$$|\beta\rangle = \sqrt{\frac{6}{\pi^2}} \sum_{n=1}^{\infty} \sqrt{\frac{1}{n^2}}\,|n\rangle$$

Obviously $\beta \in \mathcal{H}$ as the sequence of coefficients above is convergent. Then recalling $\|T\psi\|^2 = \langle T\psi|T\psi\rangle$, it is immediate that $\hat{n}$ is unbounded since $\|\hat{n}\beta\|$ reduces to the sum of all natural numbers and hence diverges,[2] violating the inequality (1) and the criterion '...$\forall\ \psi \in \mathcal{H}$'.

There are various subspaces related to $T \in \mathcal{H}$ which we call upon later; so we define them now:

1. $\ker(T) = \{\psi \in \mathcal{H} \mid T\psi = 0\}$ **(kernel)**
2. $\mathrm{ran}(T) = \{\psi \in \mathcal{H} \mid \psi = T\varphi\ \forall\ \varphi \in \mathcal{H}\}$ **(range)**
3. $\mathrm{supp}(T) = \{\psi \in \mathcal{H} \mid \psi \perp \phi\ \forall\ \phi \in \ker(T)\}$ **(support)**

The kernel represents the null space of T, and the support represents the subspace of non-null results. Obviously the range is the union of the support and the kernel. We call $\dim[\mathrm{supp}(T)]$ the **rank** of T.

Let us now discuss some qualities of the space of bounded operators $\mathfrak{L}(\mathcal{H})$. The set $\mathfrak{L}(\mathcal{H})$ is a vector space such that i) $S+T \in \mathfrak{L}(\mathcal{H})$ for $S, T \in \mathfrak{L}(\mathcal{H})$ *(additivity)* and ii) $cT \in \mathfrak{L}(\mathcal{H})$ for $c \in \mathbb{C}, T \in \mathfrak{L}(\mathcal{H})$ *(scaling)*. More importantly, $\mathfrak{L}(\mathcal{H})$ is a normed space with a **spectral norm** (or operator norm) defined as:

$$\|T\| = \sup_{\|\psi\|=1} \|T\psi\| = \sup_{\|\psi\|=1} |\langle T\psi|T\psi\rangle| \tag{2}$$

where we have deliberately written out the norm as an overlap for pedagogical reasons. Here, sup refers to the supremum ('least upper bound') such that $\|T\|$ is the least number $t \geq 0$ satisfying $\|T\psi\| \leq t\|\psi\|$. To show that $\mathfrak{L}(\mathcal{H})$ is indeed a normed space, one must check that definition (2) satisfies the properties of normed spaces listed in lecture [ref lect1] (Hint: it does!). Furthermore, one can show that $\mathfrak{L}(\mathcal{H})$ is *complete* in the operator norm (again, see Lecture [ref lect1] for discussion on completeness).

One useful property of the operator norm is that: given $T \in \mathfrak{L}(\mathcal{H})$, then $\forall\ \psi \in \mathcal{H} \implies \|T\psi\| \leq \|T\| \cdot \|\psi\| \implies |\langle\varphi|T\psi\rangle| \leq \|\varphi\| \cdot \|\psi\| \cdot \|T\|$ for arbitrary $\varphi, \psi \in \mathcal{H}$. This sets upper bounds on the overlap $|\langle\varphi|T\psi\rangle|$ for arbitrary states of the Hilbert space. Using this intuition, we can actually provide upper bounds for *products* of operators as well. Given $S, T \in \mathfrak{L}(\mathcal{H}) \implies \|S \cdot T\|$ is bounded via:

$$\|S \cdot T\psi\| \leq \|S\| \cdot \|T\| \cdot \|\psi\| \implies \|S \cdot T\| \leq \|S\| \cdot \|T\|$$

which follows because $S, T \in \mathfrak{L}(\mathcal{H})$ and thus $\|S\|, \|T\|$ are bounded. Products of bounded operators are therefore bounded operators. This suggests a notion of multiplication for the set $\mathfrak{L}(\mathcal{H})$, implying further that $\mathfrak{L}(\mathcal{H})$ has an algebraic structure. In particular we will see later that $\mathfrak{L}(\mathcal{H})$ is a C^*-*algebra*. To show this, let us first define the **adjoint** $T^\dagger$ for $T \in \mathfrak{L}(\mathcal{H})$ through:

$$\langle\varphi|T\psi\rangle = \left\langle T^\dagger\varphi \middle| \psi \right\rangle \tag{3}$$

[2] Or equates to $-\frac{1}{12}\ldots$

$\forall\ \psi, \varphi \in \mathcal{H}$. The map $T \to T^\dagger$ is **conjugate linear** such that: i) $(cS+T)^\dagger = \overline{c}S^\dagger + T^\dagger$ for $c \in \mathbb{C}$ and ii) $(ST)^\dagger = T^\dagger S^\dagger$.

Proposition 1. *A bounded operator $T \in \mathfrak{L}(\mathcal{H})$ and its adjoint $T^\dagger$ satisfy:*

$$\|T\| = \left\|T^\dagger\right\| = \left\|T \cdot T^\dagger\right\|^{\frac{1}{2}} = \left\|T^\dagger \cdot T\right\|^{\frac{1}{2}} \tag{4}$$

Proof. Let $\psi \in \mathcal{H}$ such that $\|\psi\| = 1$ and consider the following:

$$\begin{aligned} &\|T\psi\|^2 = |\langle T\psi|T\psi\rangle| = \left|\left\langle \psi\middle|T^\dagger T\psi\right\rangle\right| \leq \|\psi\|^2 \left\|T^\dagger T\right\| = \left\|T^\dagger T\right\| \leq \left\|T^\dagger\right\| \cdot \|T\| \\ &\implies \|T\|^2 \leq \left\|T^\dagger T\right\| \leq \left\|T^\dagger\right\| \cdot \|T\| \end{aligned}$$

But the substitutions $T \to T^\dagger$ and $T^\dagger \to T$ also implies:

$$\left\|T^\dagger\right\|^2 \leq \left\|T \cdot T^\dagger\right\| \leq \|T\| \cdot \left\|T^\dagger\right\|$$

When taken together, the last two implications suggest the proposition. □

Collecting the properties above, we conclude that $\mathfrak{L}(\mathcal{H})$ is a C^***-algebra** such that:

1. $\mathfrak{L}(\mathcal{H})$ is an algebra
2. $\mathfrak{L}(\mathcal{H})$ is a complete normed space (so-called *Banach space*)
3. the adjoint mapping is conjugate liner as defined above
4. the operator norm satisfies:
   (a) $\|ST\| \leq \|S\| \cdot \|T\|$ (sub-multiplicativity)
   (b) Proposition 1

Let us consider an example bounded operator, the 'shift operator', which we will use as an example multiple times through these lectures.

Example 2. *Write $\zeta \in l^2(\mathbb{N})$ as $\zeta = \langle \zeta_0, \zeta_1, \dots \rangle$. The* ***shift operator*** *is then defined via:*

$$A\langle \zeta_0, \zeta_1, \dots \rangle = \langle 0, \zeta_0, \zeta_1, \dots \rangle \tag{5}$$

A is then bounded because $\|A\zeta\| = \|h\| \implies \|A\| = 1$.

<u>Remark</u>: The adjoint of the shift operator $A^\dagger$ is defined via:

$$A^\dagger\langle \zeta_0, \zeta_1, \dots \rangle = \langle \zeta_1, \zeta_2, \dots \rangle \tag{6}$$

Note that the adjoint $A^\dagger$ (or 'left shift' operator) does not have a left inverse as the first entry is irrevocably lost in the shift process. However, also note that $A^\dagger$ is the left inverse of A.

In a finite dimensional Hilbert space $\mathcal{H}$, the set of bounded operators $\mathfrak{L}(\mathcal{H})$ consists of *all* linear mappings and is identified with all $d \times d$ matrices with complex entries, $M_d(\mathbb{C})$. This implies that one can fix an orthonormal basis $\{\varphi_j\}_{j=1}^d$ such that for $T \in \mathfrak{L}(\mathcal{H})$ [i.e. $T \in M_d(\mathbb{C})$]:

$$\begin{aligned} T_{jk} &= \langle \varphi_j | T | \varphi_k \rangle \\ \implies T |\psi\rangle &= \sum_{j,k}^{d} T_{jk} \langle \varphi_k | \psi \rangle |\varphi_j\rangle \end{aligned} \tag{7}$$

where $\psi \in \mathcal{H}$ and T_{jk} is the *matrix representation* of operator T. If $\dim \mathcal{H} = d \to \infty$, the procedure is analogous, i.e. T_{jk} is defined as above but with respect to an infinite dimensional orthonormal basis $\{\varphi_j\}_{j=1}^\infty$.

Proposition 2. *$S = T$ iff $\langle \psi | S\psi \rangle = \langle \psi | T\psi \rangle \ \forall \ \psi \in \mathcal{H}$.*

Proof. Suppose that $\langle \psi | S\psi \rangle = \langle \psi | T\psi \rangle$. Now recall the **Polarization identity**:

$$\langle \phi | T\psi \rangle = \frac{1}{4} \sum_{k=0}^{3} i^k \left\langle \psi + i^k \phi \middle| T(\psi + i^k \phi) \right\rangle \tag{8}$$

which holds for arbitrary $\phi, \psi \in \mathcal{H}$. Our supposition together with the polarization identity then suggests that $\langle \phi | S\psi \rangle = \langle \phi | T\psi \rangle \ \forall \ \psi, \phi \in \mathcal{H} \implies S |\psi\rangle = T |\psi\rangle$ generically for arbitrary ψ. From the perspective of matrix representations, this implies that all matrix elements of S are equal to those of T, element by element, via relation (7) (infinite criteria to check!). Therefore, $S = T$. □

We move on to discuss eigenvalues of bounded operators. Let $T \in \mathfrak{L}(\mathcal{H})$ and $\lambda \in \mathbb{C}$. Then:

1. λ is an (non-zero) **eigenvalue** of T if $T\psi = \lambda\psi$ where ψ is an **eigenvector** of T
2. λ is in the **spectrum** of T if the inverse of $T - \lambda\mathbb{I}$ does not exist, with $\mathbb{I}$ the identity

In finite dimensions, every operator has eigenvalues $\{\lambda\}$ which satisfy the equation $\det(T - \lambda\mathbb{I})$. However this is is not generically true in infinite dimensions. Though such an operator *will* have a spectrum.

Example 3. *The (right) shift operator A does not have eigenvalues. For suppose it did, then $A\psi = \lambda\psi$. Let $\psi = c_k\delta_k$ where $c_k \in \mathbb{C}$ and $\{\delta_k\}_{k=0}^\infty$ the set of Kronecker functions of $l^2(\mathbb{N})$. Note that $A\delta_k = \delta_{k+1}$ by definition of the shift. Then:*

$$A\psi = \lambda \sum_{k=0}^{\infty} c_k \delta_k = \sum_{k=0}^{\infty} c_k \delta_{k+1} = \sum_{k=1}^{\infty} c_{k-1} \delta_k$$

The above implies that $\lambda c_0 = 0$. But $\lambda c_1 = c_0$ and so on down the line. This holds only for $c_k = 0 \ \forall \ k \implies \psi = 0$. Note however that A has a spectrum, for $\lambda = 0$ satisfies the above and $A - 0 \cdot \mathbb{I} = A$ is non-invertible[3]*. We say that $\lambda = 0$ is* in the spectrum *of A.*

[3] A has a left inverse $A^\dagger$ such that $A^\dagger A = \mathbb{I}$. However, a right inverse for A does not exist. For if it did then it would be $A^\dagger$ such that $A \cdot A^\dagger = \mathbb{I}$. From the definition of $A^\dagger$ (6), we see that this cannot be true in general since ζ_0 is irrevocably lost when we first 'shift left' with $A^\dagger$ and then try to going back by 'shifting right' with A. A needs both right *and* left inverses to be called invertible.

Let us now discuss briefly a special class of operators, self-adjoint operators. As mentioned in the overview 1, this is an important class of operators as e.g. the set of all observables consist of self-adjoint operators via the second postulate of quantum theory. So let us unambiguously define these objects now. A bounded operator $T \in \mathfrak{L}(\mathcal{H})$ is **self-adjoint** if $T = T^\dagger$. Let $\mathfrak{L}_s(\mathcal{H})$ denote the set of all bounded self-adjoint operators. Note that $\mathfrak{L}_s(\mathcal{H})$ is a *real* vector space because all linear combinations of self-adjoint operators with real coefficients are also self-adjoint operators.

Proposition 3. $T \in \mathfrak{L}_s(\mathcal{H})$ *iff* $\langle \psi | T\psi \rangle \in \mathbb{R} \ \forall \ \psi \in \mathcal{H}$.

The proof of this proposition is in most basic/intermediate textbooks on quantum mechanics when e.g. the author(s) discuss Hermitian (self-adjoint) operators. So we will not provide the explicit proof here (exercise!).

Claim 1. *We can define so-called* ***positive semi-definite*** *(PSD) operators:* $T \in \mathfrak{L}(\mathcal{H})$ *is PSD if* $\langle \psi | T\psi \rangle \geq 0 \ \forall \ \psi \in \mathcal{H}$. *By definition then, all PSD operators are self-adjoint.*