PHYS 7895: Gaussian Quantum Information

Lecture 1

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1 Overview

Separable Hilbert spaces form the backbone upon which the quantum theory in this course stems. Indeed the first postulate of quantum mechanics is the existence of such an arena where physical states reside, evolve, etc. Therefore sufficient knowledge of a Hilbert space's structure, defining properties, etc. is well worth earnest study. Hence this first lecture focuses on properly defining and discussing properties about separable Hilbert spaces. In succinct terms:

Definition 1. *A Hilbert space $\mathcal{H}$ is a complete inner product space.*

In this lecture, we first define what makes $\mathcal{H}$ an inner product (IP) space and what makes it complete. We then establish a definition of separability for an IP space. From there, we combine all these notions to unambiguously define a separable Hilbert space and discuss some of its properties.

2 Hilbert Spaces

Suppose $\mathcal{H}$ is a complex **vector space**.[1] Given $|\psi\rangle, |\varphi\rangle, |\phi\rangle \in \mathcal{H}$ and $c \in \mathbb{C}$, if there exists a map $\langle\cdot|\cdot\rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ (i.e. an **inner product**) that satisfies the following, then $\mathcal{H}$ is an inner product space:

1. $\langle\psi|c\varphi + \phi\rangle = c\langle\psi|\varphi\rangle + \langle\psi|\phi\rangle$ *(linearity)*
2. $\overline{\langle\psi|\varphi\rangle} = \langle\varphi|\psi\rangle$ *(conjugate)*
3. $\langle\psi|\psi\rangle \geq 0$, and $\langle\psi|\psi\rangle = 0$ iff $|\psi\rangle = |0\rangle =$ with $|0\rangle$ being the 'null vector' *(positivity)*

Note that if $\langle\psi|\varphi\rangle = 0$ but $|\varphi\rangle, |\psi\rangle \neq |0\rangle$ then φ, ψ are said to be **orthogonal**. Using this notion of orthogonality, one can show the following: given an inner product space $\mathcal{H}$, if *for any* positive integer d there exists an orthogonal set of d vectors in $\mathcal{H}$, then $\dim \mathcal{H} = \infty$. The proof is by induction. That is, given a set of d orthogonal vectors of $\mathcal{H}$, one can find another set of $d + 1$ orthogonal vectors of $\mathcal{H}$ (by assumption of "*any* positive integer...”), but this can be iterated infinitely many times, which proves the statement. In colloquial language "There's always room at the Hilbert hotel."[2] There are many more interesting and useful properties of inner product (IP) spaces. For example, given two IP spaces $\mathcal{H}$ and $\mathcal{H}'$, if there exists a bijective (one-one and

[1] We assume knowledge of what defines a vector space.
[2] See Hilbert Hotel on Wikipedia.

onto) linear map $U : \mathcal{H} \to \mathcal{H}'$ such that $\langle \psi' | \varphi' \rangle = \langle \psi | \varphi \rangle$, where $|\psi'\rangle = U |\psi\rangle$, then $\mathcal{H}$ and $\mathcal{H}'$ are **isomorphic** and U is an isomorphism of the IP spaces. This bears physical significance in the quantum theory regarding, e.g., the invariance of transition amplitudes, the Born rule, etc., under isomorphisms.

With discussion of generic properties of IP spaces aside, we can focus on a particular example of an IP space which is useful for our purposes. Let $\mathbb{N}$ be the set of natural numbers $\{0, 1, 2, \ldots\}$, and let $\ell^2(\mathbb{N})$ be a set of functions $f : \mathbb{N} \to \mathbb{C}$ such that:

$$\sum_{j=0}^{\infty} |f_j|^2 < \infty \tag{1}$$

with $f_j = f(j)$. The formula for the inner product of $l^2(\mathbb{N})$ is then

$$\langle g | f \rangle = \sum_{j=0}^{\infty} \overline{g}_j f_j \tag{2}$$

which satisfies all the properties (i)–(iii) above. Hence $\ell^2(\mathbb{N})$ is an IP space. A consequence of these properties is that if $f, g \in \ell^2(\mathbb{N})$ [i.e. f, g independently satisfy relation (1)] then $\langle g | f \rangle < \infty$. This follows by applying the **Cauchy-Scwharz inequality**:

$$|\langle \varphi | \psi \rangle|^2 \leq \|\varphi\|^2 \|\psi\|^2 \tag{3}$$

valid for any IP space, to elements of $\ell^2(\mathbb{N})$.

Continuing with this example, define the **Kronecker functions** $\{\delta_k\}_{k=0}^{\infty}$ on $\ell^2(\mathbb{N})$ as

$$\delta_{kj} = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{else} \end{cases} \tag{4}$$

with $\delta_{kj} = \delta_k(j)$ such that $\langle \delta_k | \delta_j \rangle = \delta_{kj}$.[3] The Kronecker functions constitute an infinite set (one for each natural number) of orthogonal vectors of $\ell^2(\mathbb{N})$. Therefore, $\ell^2(\mathbb{N})$ is an infinite dimensional IP space.

An IP space comes from the more general **normed space** with norm $\|\psi\|$. Particular to our interests, the norm can be defined through the inner product such that $\|\psi\| = \sqrt{\langle \psi | \psi \rangle}$ and satisfies the following properties. Given $\psi, \varphi \in \mathcal{H}$ and $c \in \mathbb{C}$:

1. $\|\psi\| \geq 0$ and $\|\psi\| = 0$ iff $\psi = 0$
2. $\|c \cdot \psi\| = |c| \cdot \|\psi\|$
3. $\|\psi + \varphi\| \leq \|\psi\| + \|\varphi\|$ *(triangle inequality)*

One may check the validity of these properties for an inner product space by using the definition of the norm, the properties of List 3, and the Cauchy-Schwarz inequality (3). These properties also imply fundamental geometric relations such as:

[3] Usually the ket representation of Kronecker functions δ_n goes as $|n\rangle$ (e.g., the Fock states of the quantum harmonic oscillator). Both notations will be used in these notes.

1. Pythagorean theorem: $\|\psi + \varphi\|^2 = \|\psi\|^2 + \|\varphi\|^2 \quad$ (for ψ, φ orthogonal)
2. Parallelogram law: $\|\psi + \varphi\|^2 + \|\psi - \varphi\|^2 = 2 \cdot \|\psi\|^2 + 2 \cdot \|\varphi\|^2$

The norm induces a metric (in the topological sense) on $\mathcal{H}$, making $\mathcal{H}$ a metric space,[4] such that the following **distance measure** may be defined for elements $\psi, \varphi \in \mathcal{H}$:

$$d(\psi, \varphi) = \|\psi - \varphi\| \tag{5}$$

One can then use the distance to show that the inner product $|\langle\psi|\varphi\rangle|$ is continuous. Indeed, define $\epsilon = \|\varphi_2 - \varphi_1\| > 0$ and use the Cauchy-Schwarz inequality to show:

$$|\langle\psi|\varphi_2 - \varphi_1\rangle| \leq \|\psi\| \cdot \epsilon \tag{6}$$

In other words, 'small' changes in φ lead to small changes in the inner product.

With prerequisite knowledge of metric spaces, IP spaces, etc., laid forth, we now discuss the notion of completeness. A metric space $\mathcal{H}$ is called **complete** if every Cauchy sequence is convergent and the limit of the sequence is within the space. Recall that a sequence $\{\varphi_j\}_j$ is Cauchy if $\forall\ \epsilon > 0\ \exists\ N_\epsilon$ such that $d(\varphi_j, \varphi_k) \leq \epsilon$ whenever $j, k \geq N_\epsilon$. To understand the completeness criterion a bit, let us consider a counterexample.

Example 1. *Consider the sequence $\varphi_n = \frac{1}{n}$ in the interval $(0, 1]$. One can prove that this sequence is Cauchy by choosing $N_\epsilon > \frac{2}{\epsilon}$ and forming the following set of inequalities:*

$$|\varphi_n - \varphi_m| \leq \frac{1}{n} + \frac{1}{m} \leq \frac{2}{N_\epsilon} < \epsilon \tag{7}$$

where we have used the criterion $m, n \geq N_\epsilon$. However, the sequence is not complete because $\lim_{n\to\infty} \varphi_n = 0 \notin (0, 1]$. However, if we close the interval $[0, 1]$, it becomes complete.

A consequence of completeness is the existence of basis expansions, where the basis is some set of **orthonormal vectors** of $\mathcal{H}$. Recall, a set of orthonormal vectors is a set of orthogonal vectors $\mathcal{X} \subset \mathcal{H}$ such that the norm of each vector $\phi \in \mathcal{X}$ is unity. The set is said to be **maximal** if there does not exist an orthonormal set which contains $\mathcal{X}$ as a proper subset. If an othornormal set $\mathcal{X}$ is maximal, then it constitutes an **orthonormal basis** of $\mathcal{H}$. We note an important property, without direct proof, that every Hilbert space $\mathcal{H}$ has an orthonormal basis $\mathcal{X}$ and *all* orthornomal bases $\{\mathcal{X}_i\}$ of $\mathcal{H}$ have the same cardinality $|\mathcal{X}_i| = \dim \mathcal{H}$. A consequence of maximality is that: if $\mathcal{X}$ is an orthonormal basis of $\mathcal{H}$ and there exists $\psi \in \mathcal{H}$ such that $\langle\psi|\phi\rangle\ \forall\ \phi \in \mathcal{X}$, then $\psi = 0$.

Separability of a space follows from here. A space $\mathcal{H}$ is **separable** if it has a countable orthonormal basis. Here, countable is defined in the usual topological sense, e.g. a set $\mathcal{H}$ has a countable orthonormal basis $\mathcal{X}$ if each element of $\mathcal{X}$ can be associated uniquely to an element of $\mathbb{N}$. As example, consider the Hilbert (i.e. complete IP) space $\ell^2(\mathbb{N})$. It has a basis given by the Kronecker functions $\{\delta_j\}\ \forall\ j \in \mathbb{N}$. The set of Kronecker functions is obviously maximal as each δ_j is associated with a natural number $j \in \mathbb{N}$, but this also implies the orthonormal basis is countable. Hence $\ell^2(\mathbb{N})$ *is a separable Hilbert space.* Indeed it is *the* separable Hilbert space.

Proposition 1. *Any separable Hilbert space $\mathcal{H}$ is isomorphic to $\ell^2(\mathbb{N})$.*

[4]There is a hierarchy of generality, metric space→ normed space→ vector space, from more general to less general. Hence properties applying to metric spaces apply to normed spaces etc.

Proof. Fix the an orthonormal basis $\{\varphi_k\}_{k=0}^{\infty}$ for some separable Hilbert space $\mathcal{H}$. For each $\psi \in \mathcal{H}$, define $f : \mathbb{N} \to \mathbb{C}$ via $f_j = \langle \varphi_j | \psi \rangle$, then $f \in \ell^2(\mathbb{N})$ and the map $\psi \to f$ is an isomorphism between $\mathcal{H}$ and $\ell^2(\mathbb{N})$. $\square$