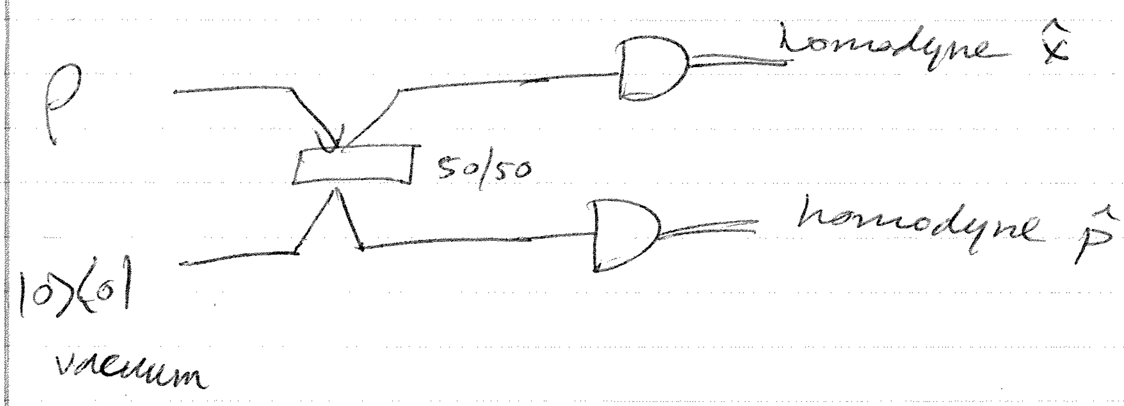
Lecture 24

Continuing w/ Gaussian measurements + moving on to heterodyne detection:

Recall that coherent states form a resolution of identity:

$$\frac{1}{\pi}\int d^2\alpha \; |\alpha\rangle\langle\alpha| = \hat{I}$$

Heterodyne detection can be performed (approximately) by the following experimental setup:



Why does this work?

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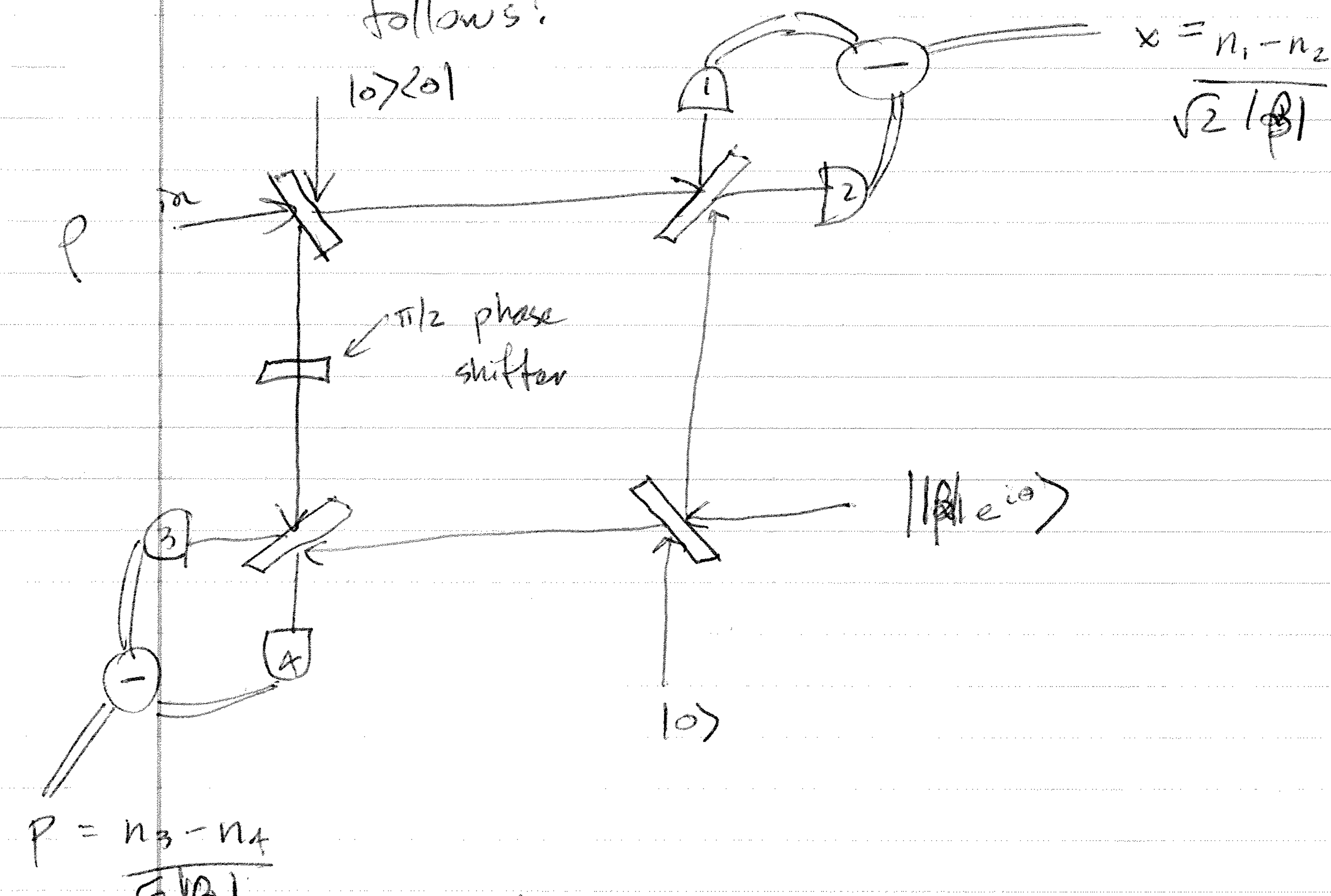
More specifically, this can be done as follows:



All beamsplitters are 50:50

This realizes projection onto $|\alpha\rangle = \left|\frac{x+ip}{\sqrt{2}}\right\rangle$

Consider that, by defining

$$|\Gamma\rangle_{SP} = \sum_{n=0}^{\infty} |n\rangle_S |n\rangle_P ,$$ & observing that

$\langle 0|_P |\Gamma\rangle_{SP} = |0\rangle_S$

we find that

$$\frac{1}{\pi}\langle\alpha|\rho|\alpha\rangle = \frac{1}{\pi}\langle 0|\hat{D}_\alpha^\dagger \rho \hat{D}_\alpha|0\rangle$$

$$= \frac{1}{\pi}\langle 0|_S \left(\hat{D}_\alpha^\dagger\right)_S \rho_S \left(\hat{D}_\alpha\right)_S |0\rangle_S$$

$$= \frac{1}{\pi}\langle\Gamma|_{SP}|0\rangle_P \left(\hat{D}_r\right)_S \rho_S \left(\hat{D}_{-r}\right)_S \langle 0|_P |\Gamma\rangle_{SP}$$

$$= \frac{1}{\pi}\langle\Gamma|_{SP}\left(\hat{D}_r\right)_S \left(\rho_S \otimes |0\rangle\langle 0|_P\right)\left(\hat{D}_{-r}\right)_S |\Gamma\rangle_{SP}$$

Thus, the projection onto a coherent state is equivalent to tensoring in vacuum state & projecting onto ket

$$\frac{1}{\sqrt{\pi}}\left(\hat{D}_{-r}\right)_S |\Gamma\rangle_{SP}$$

Claim is that both ideal homodynes implement this projection.

Ideal general-dyne detection

3

coherent-state resolution of identity extends to many modes as

$$\hat{I} = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{2n}} d\underline{r} \; \hat{D}_{-\underline{r}} |0\rangle\langle 0| \hat{D}_{\underline{r}}$$

can then get resolution as

$$\hat{I} = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{2n}} d\underline{r} \; \hat{S}^{+} \hat{D}_{-\underline{r}} |0\rangle\langle 0| \hat{D}_{\underline{r}} \hat{S}$$

$$= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{2n}} dr \; \hat{D}_{-\underline{r}} \hat{S}^{+} |0\rangle\langle 0| \hat{S} \hat{D}_{\underline{r}}$$

for $\hat{S}$ quadratic w/ symplectic S.

for the latter equality, use that $\hat{S}^{+} \hat{D}_r S = \hat{D}_{S\underline{r}}$,

change of variable to $S\underline{r}$

& use that $\mathrm{Det}(S) = 1$ for symplectic measurement process corresponds to projection onto pure Gaussian state

$$|\psi_G\rangle = \hat{D}_{-\underline{r}}\, \hat{S}^{+} |0\rangle$$

$\Rightarrow$ probability density is given by

$$\frac{1}{(2\pi)^n} \langle \psi_G | \rho | \psi_G \rangle$$

heterodyne is recovered for choice $\hat{S} = \hat{I}$

while homodyne is recovered for $\hat{S}$ an infinite squeezer, i.e.,

$$S = R_\varphi^T \begin{bmatrix} z & 0 \\ 0 & 1/z \end{bmatrix} R_\varphi$$

w/ $z \to 0$ + $R_\varphi = \begin{bmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{bmatrix}$

selects the phase for homodyne

Suppose that the measured state ρ is Gaussian w/ mean $\bar{r}$ & cov. matrix σ. Then probability density for ideal general-dyne is given by

$$p(\underline{r}) = \frac{1}{(2\pi)^n} \langle \psi_\sigma | \rho | \psi_\sigma \rangle$$

$$= \frac{e^{-(\underline{r}-\bar{r})^T (\sigma + SS^T)^{-1} (\underline{r}-\bar{r})}}{\pi^n \sqrt{\mathrm{Det}(\sigma + SS^T)}}$$

can understand single-mode homodyne limit by taking $\lim_{z \to \infty}$ w/ $S = \begin{bmatrix} z & 0 \\ 0 & 1/z \end{bmatrix}$

Consider that

$$\lim_{z \to 0} (\sigma + SS^T)^{-1} = \begin{bmatrix} \sigma_{11}^{-1} & 0 \\ 0 & 0 \end{bmatrix}$$

Also note that integrating out

p variable gives

$$\frac{e^{-(x-\bar{x})^2/(\sigma_{11}+z^2)}}{\sqrt{\pi(\sigma_{11}+z^2)}}$$

so that the above →

$$\frac{e^{-(x-\bar{x})^2/\sigma_{11}}}{\sqrt{\pi\sigma_{11}}} \quad \text{as } z \to 0.$$

squeezing

variance of the distribution is

$\sigma_{11}/2$ w/ extra factor of 2 coming from definition of cov. matrix

Previously considered the case of ideal general-dyne measurements. Now we consider the case of noisy general-dyne measurements.

Consider that a unital, completely positive map preserves the identity, i.e., $\mathcal{N}^{\dagger}(I) = I$ - this means that

$$I = \mathcal{N}^{\dagger}(I) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{2n}} d\underline{r} \; \mathcal{N}^{\dagger}\left(\hat{D}_{-\underline{r}} |0\rangle\langle 0| \hat{D}_{\underline{r}}\right)$$

is a resolution of the identity. This meas. is equivalent to 1st acting w/ $\mathcal{N}$ on state & then doing ideal heterodyne

The dual of any q. Gaussian channel is Gaussian & unital.

Outcome probability is then

$$p(\underline{r}) = \frac{1}{(2\pi)^n} \operatorname{Tr}\left[|\psi_G\rangle\langle\psi_G| \, \mathcal{N}(\rho)\right]$$

$$= \frac{1}{(2\pi)^n} \operatorname{Tr}\left\{\mathcal{N}^\dagger(|\psi_G\rangle\langle\psi_G|)\,\rho\right]$$

this motivates the need to derive the action of the adjoint map $\mathcal{N}^\dagger$ on a Gaussian state.

Previously, we showed that a q. Gaussian channel $\mathcal{N}$ w/ X + Y matrices is such that

$$\mathcal{N}^\dagger(\hat{D}_{\Omega \underline{r}}) = \hat{D}_{\Omega X^T \underline{r}} \, e^{-\frac{1}{4}\underline{r}^T Y \underline{r}}$$

we can then use this to determine the action of $\mathcal{N}^\dagger$ on a general Gaussian state.

9

Recalling that a general Gaussian state can be written as

$$\rho_G = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{2n}} d\tilde{\underline{r}}\; e^{-\frac{1}{4}\tilde{\underline{r}}^T \sigma \tilde{\underline{r}} + i\tilde{\underline{r}}^T \bar{r}}\; \hat{D}_{\Omega^T \tilde{\underline{r}}}$$

where mean of ρ_G is $\bar{r}$ & cov. matrix is σ.

Then we find that

$$\mathcal{N}^\dagger(\rho_G) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{2n}} d\tilde{\underline{r}}\; e^{-\frac{1}{4}\tilde{\underline{r}}^T \sigma \tilde{\underline{r}} + i\tilde{\underline{r}}^T \bar{r}}\; \underbrace{\mathcal{N}^\dagger(\hat{D}_{\Omega^T \tilde{\underline{r}}})}_{=\hat{D}_{\Omega(-\tilde{\underline{r}})}}$$

$$= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{2n}} d\tilde{\underline{r}}\; e^{-\frac{1}{4}\tilde{\underline{r}}^T \sigma \tilde{\underline{r}} + i\tilde{\underline{r}}^T \bar{r}}\; e^{-\frac{1}{4}\tilde{\underline{r}}^T Y \tilde{\underline{r}}}\; \hat{D}_{\Omega X^T(-\tilde{\underline{r}})}$$

$$= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{2n}} d\tilde{\underline{r}}\; e^{-\frac{1}{4}\tilde{\underline{r}}^T (\sigma + Y) \tilde{\underline{r}} + i\tilde{\underline{r}}^T \bar{r}}\; \hat{D}_{\Omega^T X^T \tilde{\underline{r}}}$$

if X is invertible, then make the substitution $\underline{r}' = X^T \tilde{\underline{r}}$

$$\Rightarrow \quad \tilde{\underline{r}} = X^{-T} \underline{r}'$$

& we get that

$\mathcal{N}^{+}(\rho_G)$

$$= \frac{1}{|\mathrm{Det}(X)|} \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{2n}} e^{-\frac{1}{4} \underline{r}'^T (X^{-1}[\sigma + Y] X^{-T}) \underline{r}' + i \underline{r}'^T X^{-1} \underline{r}} \hat{D}_{\Omega^T \underline{r}'}$$

this implies that the action is given by

$$X_{dual} = X^{-1}$$

$$Y_{dual} = X^{-1} Y X^{-T}$$

w/ $\frac{1}{|\mathrm{Det}(X)|}$ multiplying the input density operator.

So a Gaussian CP map w/ invertible X is unital iff $|\mathrm{Det}(X)| = 1$

Let's now return to noisy general dyne measurements. We find that

$$p(\underline{r}) = \frac{1}{(2\pi)^n} \mathrm{Tr}\left[\mathcal{N}^\dagger(|\psi_G\rangle\langle\psi_G|)\, \rho \right]$$

& $\mathcal{N}^\dagger(|\psi_G\rangle\langle\psi_G|)$

$$= \frac{\hat{D}_{-X^{-1}\underline{r}}\, \rho_m\, \hat{D}_{X^{-1}\underline{r}}}{|\mathrm{Det}(X)|}$$

where ρ_m is a G. state w/ mean zero + cov. matrix

$$\sigma_m = X^{-1} S S^T X^{-T} + X^{-1} Y X^{-T}$$

We can then multiply the measurement outcome $\underline{r}$ by X

& then the probability density gets scaled by $|\mathrm{Det}(X)|$

Recalling that any Gaussian state covariance matrix can be written as $\sigma = SS^T + Y$ where S is symp. matrix $\& \ Y \geq 0$, by taking $X = I$, we find that the noisy general-dyne detection realizes the measurement

$$\left\{ \frac{1}{(2\pi)^n} \hat{D}_{-\underline{r}_m} \rho_m \hat{D}_{\underline{r}_m} \right\}_{r_m}$$

where ρ_m is a generic Gaussian state w/ mean zero & covariance matrix σ_m s.t. $\sigma_m + i\Omega \geq 0$

Then $\hat{I} = \int_{\mathbb{R}^{2n}} d\underline{r}_m \frac{1}{(2\pi)^n} \hat{D}_{-\underline{r}_m} \rho_m \hat{D}_{\underline{r}_m}$

& prob. density is

$$p(\underline{r}_m) = \mathrm{Tr}\{\rho \hat{D}_{-r_m} \rho_m \hat{D}_{\underline{r}_m}\} / (2\pi)^n$$

If the measurement is performed on a Gaussian state w/ mean $\bar{r}$ + cov. matrix σ, then prob. density is given by

$$p(\underline{r}_m) = \frac{e^{-(\underline{r}_m-\bar{r})^T(\sigma+\sigma_m)^{-1}(\underline{r}_m-\bar{r})}}{\pi^n\sqrt{\mathrm{Det}(\sigma+\sigma_m)}}$$

Conditional dynamics

Suppose G. meas. is performed on one share B of a G. state ρ_{AB}.

What is prob. of outcome + what is post meas. state?

~~Need~~ Suppose ρ_{AB} has mean $\begin{bmatrix}\bar{r}_A\\ \bar{r}_B\end{bmatrix}$

+ cov. matrix $\begin{bmatrix}\sigma_A & \sigma_{AB}\\ \sigma_{AB}^T & \sigma_B\end{bmatrix}$

Suppose general-dyne characterized by

$$\tau_B^{r_m} = \frac{1}{(2\pi)^{n_B}} \hat{D}_{-\underline{r}_m} \rho_m \hat{D}_{\underline{r}_m} \quad \text{for mode } B$$

then we want to calculate

$$\mathrm{Tr}_B \left[\rho_{AB} \tau_B^{\underline{r}} \right] \qquad n = n_A + n_B$$

then write

$$\rho_{AB} = \frac{1}{(2\pi)^{n_A+n_B}} \int_{\mathbb{R}^{2(n_A+n_B)}} d\underline{\tilde{r}}\, e^{-\frac{1}{4}\underline{\tilde{r}}^T \sigma \underline{\tilde{r}} + i \underline{\tilde{r}}^T \underline{r}} \underbrace{\hat{D}_{\Lambda^T \underline{\tilde{r}}}}_{\hat{D}_{\Lambda^T \underline{\tilde{r}}_A} \otimes \hat{D}_{\Lambda^T \tilde{r}_B}}$$

resized Λ for A & B

then this becomes

$$\mathrm{Tr}_B \left[\rho_{AB} \tau_B^{\underline{r}} \right] =$$

$$\frac{1}{(2\pi)^n} \int_{\mathbb{R}^{2n}} d\underline{\tilde{r}}\, e^{-\frac{1}{4}\underline{\tilde{r}}^T \sigma \underline{\tilde{r}} + i \underline{\tilde{r}}^T \underline{r}}\, \hat{D}_{\Lambda^T \tilde{r}_A} \underbrace{\mathrm{Tr}\left[\hat{D}_{\Lambda^T \tilde{r}_B} \tau_B^{\underline{r}} \right]}_{\text{this is characteristic function of } \tau_B^{\underline{r}}}$$

$$\mathrm{Tr}\left[\hat{D}_{\Omega^T \tilde{r}_B} \tau^{\underline{r}}_{B}\right]$$

$$= \frac{1}{(2\pi)^{n_B}} e^{-\frac{1}{4}\tilde{r}_B^T \sigma_m \tilde{r}_B - i\tilde{r}_B^T \underline{r}_m}$$

$\Rightarrow$ overlap is

$$\frac{1}{(2\pi)^n \cdot (2\pi)^{n_B}} \int_{\mathbb{R}^{2n}} d\underline{\tilde{r}} \; e^{-\frac{1}{4}\underline{\tilde{r}}^T \sigma \underline{\tilde{r}} + i\underline{\tilde{r}}^T \underline{\bar{r}}} \; \hat{D}_{\Omega^T \tilde{r}_A} \cdot$$

$$e^{-\frac{1}{4}\tilde{r}_B^T \sigma_m \tilde{r}_B - i\tilde{r}_B^T \underline{r}_m}$$

Now use that $\underline{\tilde{r}}^T \sigma \underline{\tilde{r}} =$

$$\tilde{r}_A^T \sigma_A \tilde{r}_A + \tilde{r}_B^T \sigma_{AB}^T \tilde{r}_A + \tilde{r}_A^T \sigma_{AB} \tilde{r}_B + \tilde{r}_B^T \sigma_B \tilde{r}_B$$

$\&$ $\underline{\tilde{r}}^T \bar{r} = \tilde{r}_A^T \bar{r}_A + \tilde{r}_B^T \bar{r}_B$