Lecture 23

Gaussian measurements

- includes well known schemes such as homodyne & heterodyne detection, as well as other measurements
- those that ~~lead~~ to Gaussian distributions when performed on Gaussian states, as well as Gaussian post-measurement states when performed on Gaussian states.

Begin w/ homodyne detection:

Ideal homodyne detection consists of measurement of quadrature operator $\hat{x}_\varphi = \cos\varphi\, \hat{x} + \sin\varphi\, \hat{p}$

& has outcome probabilities

$$p(x_\varphi) = \langle x_\varphi | \rho | x_\varphi \rangle$$
$$= \mathrm{Tr}\left[|x_\varphi\rangle\langle x_\varphi| \, \rho \right]$$

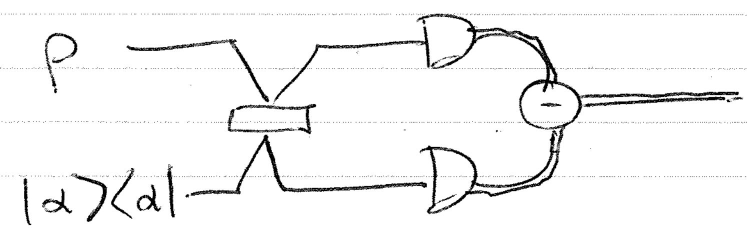
where $|x_\varphi\rangle$ is an improper eigenvector of $\hat{x}_\varphi$.

called "quadrature" or "homodyne" measurement.

Implementation:

mix ρ & strong coherent state @ 50:50 beamsplitter & subtract detected intensities @ output (using ideal photodetection)

"homodyne" - measured field mixed w/ probe of same frequency



(2a)

measurement outcome is to take photon number $\overset{\text{meas. outcome}}{n_A}$ from 1st mode

& photon # meas. outcome n_B from 2nd mode

& compute

$$\frac{n_A - n_B}{\sqrt{2}\,|\alpha|}$$

& report this as outcome of homodyne detection

in practice, we don't exactly measure photon number difference but instead measure intensity difference.

1st give heuristic justification for why this works.

Consider that $\hat{x}_\varphi = \dfrac{e^{-i\varphi}\hat{a} + e^{i\varphi}\hat{a}^\dagger}{\sqrt{2}}$

Let $\hat{b}$ be annihilation op. associated w/ auxiliary mode, prepared in coherent state.

First, let us show that expectation of measurement outcome coincides w/ that of $\hat{x}_\varphi$ in strong local oscillator limit & w/ an appropriate rescaling

Consider that expectation is given by

$$\mathrm{Tr}\left[\left(\hat{n}_A \otimes \hat{I}_B - \hat{I}_A \otimes \hat{n}_B\right) \mathcal{U}_{AB}\left(\rho \otimes |\alpha\rangle\langle\alpha|_B\right)\mathcal{U}_{AB}^\dagger\right]$$

where $\mathcal{U}_{AB}$ is unitary for 50:50 beam splitter.

~~Then this~~ Then this is equal to

$$\mathrm{Tr}\Big[\underbrace{\langle\alpha|_B \, U^\dagger_{AB} (\hat{n}_A \otimes \hat{I}_B - \hat{I}_A \otimes \hat{n}_B) \, U_{AB} |\alpha\rangle_B}_{\tilde{\hat{x}}_\varphi} \, \rho \Big]$$

$\tilde{\hat{x}}_\varphi$ ← approximation

Now use that

$$U^\dagger_{AB} (\hat{n}_A \otimes \hat{I}_B) U_{AB}$$

$$= U^\dagger_{AB} (\hat{a}^\dagger \hat{a}_A \otimes \hat{I}_B) U_{AB}$$

$$= \underbrace{U^\dagger_{AB} (a^\dagger_A \otimes I_B) U_{AB}}_{\left(\frac{\hat{a}+\hat{b}}{\sqrt{2}}\right)^\dagger} \; \underbrace{U^\dagger_{AB} (\hat{a}_A \otimes \hat{I}_B) U_{AB}}_{\frac{\hat{a}_A \otimes I_B + I_A \otimes \hat{b}_B}{\sqrt{2}}}$$

abbr. as $\left(\frac{\hat{a} + \hat{b}}{\sqrt{2}}\right)$

+ similarly

$$U^\dagger_{AB} (\hat{I}_A \otimes \hat{n}_B) U_{AB} = \left(\frac{\hat{a} - \hat{b}}{\sqrt{2}}\right)^\dagger \left(\frac{\hat{a} - \hat{b}}{\sqrt{2}}\right)$$

Then

$$U_{AB}^{\dagger}\left(\hat{n}_A - \hat{n}_B\right) U_{AB}$$

$$= \left(\frac{\hat{a}+\hat{b}}{\sqrt{2}}\right)^{\dagger}\left(\frac{\hat{a}+\hat{b}}{\sqrt{2}}\right) - \left(\frac{\hat{a}-\hat{b}}{\sqrt{2}}\right)^{\dagger}\left(\frac{\hat{a}-\hat{b}}{\sqrt{2}}\right)$$

$$= \hat{a}^{\dagger}\hat{b} + \hat{a}\,\hat{b}^{\dagger}$$

So then trace expression reduces to

$$\mathrm{Tr}\left[\langle\alpha|_B \left(\hat{a}^{\dagger}\hat{b} + \hat{a}\hat{b}^{\dagger}\right)|\alpha\rangle_B \; \rho\right]$$

Consider that

$$\hat{b}^{\dagger}|\alpha\rangle = \hat{b}^{\dagger}\hat{D}_\alpha|0\rangle_B$$

$$= \hat{D}_\alpha \hat{D}_\alpha^{\dagger}\,\hat{b}^{\dagger}\hat{D}_\alpha|0\rangle_B$$

$$= \hat{D}_\alpha\left(\hat{b}^{\dagger} + \alpha^*\right)|0\rangle_B$$

$$= \hat{D}_\alpha|1\rangle_B + \alpha^*|\alpha\rangle_B$$

operator in

$\Rightarrow$ trace expression becomes

$$\langle\alpha|_B \; \hat{a}_A^\dagger \hat{b}_B + \hat{a}_A \hat{b}_B^\dagger |\alpha\rangle_B$$

$$= \langle\alpha|_B \hat{a}_A^\dagger \hat{b}_B|\alpha\rangle_B + \langle\alpha|_B \hat{a}_A \hat{b}_B^\dagger|\alpha\rangle_B$$

$$= \left(\langle 1|_B (\hat{D}_\alpha)_B^\dagger + \alpha\langle\alpha|_B\right) \hat{a}_A^\dagger |\alpha\rangle_B$$

$$+ \langle\alpha|_B \hat{a}_A \left((\hat{D}_\alpha)_B|1\rangle_B + \alpha^*|\alpha\rangle_B\right)$$

$$= \cancel{\;}\ \langle 1|_B (\hat{D}_\alpha^\dagger)_B |\alpha\rangle_B \; \hat{a}^\dagger$$

$$+ \alpha \; \langle\alpha|\alpha\rangle_B \; \hat{a}_A^\dagger$$

$$+ \hat{a}_A \; \langle\alpha|_B (\hat{D}_\alpha)_B |1\rangle_B$$

$$+ \alpha^* \; \hat{a}_A \; \langle\alpha|\alpha\rangle_B$$

$$= \alpha\,\hat{a}_A^\dagger + \alpha^*\hat{a}_A$$

$$= |\alpha|\, e^{i\varphi}\hat{a}^\dagger + |\alpha| e^{-i\varphi}\hat{a}$$

$$= \sqrt{2}|\alpha| \; \frac{e^{i\varphi}\hat{a}^\dagger + e^{-i\varphi}\hat{a}}{\sqrt{2}} = \sqrt{2}|\alpha|\,\hat{x}_\varphi$$

So divide value of
measurement outcome by
$\sqrt{2}\,|\alpha|$ to get consistency.

Interestingly, mean value
coincides w/o any need for
strong local oscillator limit.

What about higher moments?

If all higher moments agree,
then distributions are the same

So we take approximation to
be

$$\frac{\hat{a}^\dagger \hat{b} + \hat{a}\,b^\dagger}{\sqrt{2}\,|\alpha|} = \hat{O}(|\alpha|)$$

Higher moments are then given by

$$\mathrm{Tr}\left[\hat{O}(|\alpha|)^n \left(\rho \otimes |\alpha\rangle\langle\alpha|\right)\right]$$

Defining $f(t) = \mathrm{Tr}\left[e^{it\hat{O}(|\alpha|)} \left(\rho \otimes |\alpha\rangle\langle\alpha|\right)\right]$

for ~~imaginary (purely imaginary)~~ $t \in \mathbb{R}$

Consider that

$$\left.\frac{\partial^n f(t)}{\partial t^n}\right|_{t=0} = \mathrm{Tr}\left[\hat{O}(|\alpha|)^n \left(\rho \otimes |\alpha\rangle\langle\alpha|\right)\right] \cdot (-i)^n$$

Consider 2nd moment:

$$\mathrm{Tr}\left[\hat{O}(|\alpha|)^2 \left(\rho \otimes |\alpha\rangle\langle\alpha|\right)\right]$$

$$= \frac{1}{2|\alpha|^2} \mathrm{Tr}\left[\left(\hat{a}^\dagger \hat{b} + \hat{a}\hat{b}^\dagger\right)^2 \left(\rho \otimes |\alpha\rangle\langle\alpha|\right)\right]$$

Consider that $\left(\hat{a}^\dagger \hat{b} + \hat{a}\hat{b}^\dagger\right)^2$

$$= \left(\hat{a}^\dagger \hat{b} + \hat{a}\hat{b}^\dagger\right)\left(\hat{a}^\dagger \hat{b} + \hat{a}\hat{b}^\dagger\right)$$

$$= (\hat{a}^\dagger)^2 (\hat{b})^2 + \hat{a}\hat{a}^\dagger \hat{b}^\dagger \hat{b} + \hat{a}^\dagger \hat{a}\, \hat{b}\hat{b}^\dagger + \hat{a}^2 (\hat{b}^\dagger)^2$$

$$= (\hat{a}^+)^2 (\hat{b})^2 + \hat{a}\hat{a}^+ \hat{b}^+ \hat{b} + \hat{a}^+\hat{a}\, \hat{b}^+\hat{b}$$
$$+ \hat{a}^+\hat{a} \quad + \quad \hat{a}^2 (\hat{b}^+)^2$$

Now sandwich by $\langle\alpha|_B \quad |\alpha\rangle_B$

& get

$$= (\hat{a}^+)^2 \alpha^2 + \hat{a}\hat{a}^+ |\alpha|^2$$
$$+ \hat{a}^+\hat{a} |\alpha|^2 + \hat{a}^+\hat{a} + \hat{a}^2 (\alpha^*)^2$$
$$= |\alpha|^2 \left((\hat{a}^+)^2 e^{i2\varphi} + \hat{a}\hat{a}^+ + \hat{a}^+\hat{a} + \hat{a}^2 e^{-i2\varphi} \right)$$
$$+ \hat{a}^+\hat{a}$$
$$= 2|\alpha|^2 (\hat{x}_\varphi)^2 + \hat{a}^+\hat{a}$$

$$\Rightarrow \quad Tr\left[\hat{O}(|\alpha|)^2 \left(\rho \otimes |\alpha\rangle\langle\alpha|\right)\right]$$
$$= Tr\left[(\hat{x}_\varphi)^2 \rho\right] + \frac{Tr\left[\hat{a}^+\hat{a}\,\rho\right]}{2|\alpha|^2}$$
$$= Tr\left[(\hat{x}_\varphi)^2 \rho\right] + \frac{\langle \hat{n}\rangle_\rho}{2|\alpha|^2}$$

can make 2^{nd} moment match by taking $|\alpha| \to \infty$.

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can extend this argument to all higher moments. That is, goal now is to prove that, $\forall t \in \mathbb{R}$

$$\lim_{|\alpha|\to\infty} \mathrm{Tr}\left[e^{it\hat{O}(|\alpha|)} \left(\rho \otimes |\alpha\rangle\langle\alpha|_B\right)\right]$$

$$= \mathrm{Tr}\left[e^{it\hat{x}_\varphi} \rho\right]$$

Then consider

$$\langle\alpha|_B e^{it\hat{O}(|\alpha|)}|\alpha\rangle_B = \langle\alpha|_B e^{it\left(\frac{\hat{a}^+\hat{b} + \hat{a}\hat{b}^+}{\sqrt{2}|\alpha|}\right)}|\alpha\rangle$$

$$= \langle\alpha|_B e^{\mu(\hat{a}^+\hat{b} + \hat{a}\hat{b}^+)}|\alpha\rangle$$

where $\mu = \frac{it}{\sqrt{2}|\alpha|}$

Goal is then to re-order $e^{\mu(\hat{a}^+\hat{b} + \hat{a}\hat{b}^+)}$ in normal form w/ all $\hat{b}^+$'s on left and all $\hat{b}$'s on right use the fact that algebra generated by $\hat{a}\hat{b}^{+}$, $\hat{a}^{+}\hat{b}$, $\hat{b}^{+}\hat{b}-\hat{a}^{+}\hat{a}$ is closed:

$$[\hat{a}\hat{b}^{+}, \hat{a}^{+}\hat{b}] = \hat{b}^{+}\hat{b} - \hat{a}^{+}\hat{a}$$

$$[\hat{b}^{+}\hat{b}-\hat{a}^{+}\hat{a}, \hat{a}^{+}\hat{b}] = -2\hat{a}^{+}\hat{b}$$

$$[\hat{b}^{+}\hat{b}-\hat{a}^{+}\hat{a}, \hat{a}\hat{b}^{+}] = 2\hat{a}^{+}\hat{b}$$

& isomorphic to algebra generated by

$$\hat{a}\hat{b}^{+} \leftrightarrow \sigma_{+} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \frac{\sigma_x + i\sigma_y}{2}$$

$$a^{+}\hat{b} \leftrightarrow \sigma_{-} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \frac{\sigma_x - i\sigma_y}{2}$$

$$\hat{b}^{+}\hat{b} - \hat{a}^{+}\hat{a} \leftrightarrow \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Then evaluate for $\mu \in \mathbb{C}$ w/ $|\mu| e^{i\phi} = \mu$

$$e^{\mu \hat{a}\hat{b}^{+} - \mu^{*}\hat{a}^{+}\hat{b}}$$

but use two-dimensional representation

$$e^{\mu\sigma_+ - \mu^*\sigma_-} = \begin{bmatrix} \cos|\mu| & e^{i\phi}\sin|\mu| \\ -e^{i\phi}\sin|\mu| & \cos|\mu| \end{bmatrix}$$

using that

$$(\mu\sigma_+ - \mu^*\sigma_-)^2 = -|\mu|^2 I_2$$

Also have that

$$e^{\beta\sigma_+} = \begin{bmatrix} 1 & \beta \\ 0 & 1 \end{bmatrix}, \quad e^{\gamma\sigma_z} = \begin{bmatrix} e^{\gamma} & 0 \\ 0 & e^{-\gamma} \end{bmatrix}$$

$$e^{\delta\sigma_-} = \begin{bmatrix} 1 & 0 \\ \delta & 1 \end{bmatrix}$$

Then set

$$\begin{bmatrix} \cos|\mu| & e^{i\phi}\sin|\mu| \\ -e^{-i\phi}\sin|\mu| & \cos|\mu| \end{bmatrix} = \begin{bmatrix} e^{\gamma} + \beta\delta e^{-\gamma} & \beta e^{-\gamma} \\ \delta e^{-\gamma} & e^{-\gamma} \end{bmatrix}$$

$$= e^{\beta\sigma_+} e^{\gamma\sigma_z} e^{\delta\sigma_-}$$

w/ $\gamma = -\ln\cos|\mu|$

$$\beta = -\delta^* = e^{i\phi}\tan|\mu|$$

Now use inverse of isomorphism to conclude that

$$e^{\mu\hat{a}\hat{b}^+ - \mu^*\hat{a}^+ b}$$

$$= e^{e^{i\phi}\tan|\mu|\,\hat{a}\hat{b}^+}\ \cos|\mu|^{\hat{a}^+\hat{a} - \hat{b}^+\hat{b}}\ e^{-e^{-i\phi}\tan|\mu|\,\hat{a}^+\hat{b}}$$

Going back to μ being purely imaginary, we find that (picking $\phi = \pi/2$)

$${}_B\langle\alpha| e^{\mu(\hat{a}\hat{b}^+ + \hat{a}^+\hat{b})}|\alpha\rangle_B$$

$$= e^{\tanh(\mu)\alpha^*\hat{a}}\ \cosh(\mu)^{\hat{a}^+\hat{a}}\ e^{\tanh(\mu)\alpha\hat{a}^+}\ {}_B\langle\alpha|\cosh(\mu)^{-\hat{b}^+\hat{b}}|\alpha\rangle_B$$

where we used that

$$\cos(y) = \cosh(iy) \quad \text{for } y \in \mathbb{R}$$

$$i\tan(y) = \tanh(iy)$$

Evaluate last term using number-state expansion for $|\alpha\rangle_B$ as

$$|\alpha\rangle_B = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

giving that

$$\langle\alpha|_B \cosh(\mu)^{-\hat{b}^+\hat{b}} |\alpha\rangle_B$$

$$= e^{\left(|\alpha|^2 \frac{1-\cosh\mu}{\cosh\mu}\right)}$$

then operator becomes

$$e^{\tanh(\mu)\alpha^* \hat{a}} \cosh(\mu)^{\hat{a}^+\hat{a}} e^{\tanh(\mu)\alpha \hat{a}^+} e^{\left(|\alpha|^2 \frac{1-\cosh\mu}{\cosh\mu}\right)}$$

recall that $\mu = \frac{it}{\sqrt{2}|\alpha|}$

& taking limit $|\alpha| \to \infty$ gives

$$\lim_{|\alpha|\to\infty} f(t) = e^{it\frac{e^{-i\varphi}\hat{a}}{\sqrt{2}}} e^{it\frac{e^{i\varphi}\hat{a}^+}{\sqrt{2}}} e^{-\frac{(it)^2}{4}}$$

$$= e^{it(e^{-i\varphi}\hat{a} + e^{i\varphi}\hat{a}^+)/\sqrt{2}} = e^{it\hat{x}_\varphi}$$

So the conclusion is that,
for any fixed state ρ,
we have that

$$\lim_{|\alpha|\to\infty} \mathrm{Tr}\left[\langle\alpha|_B \, e^{it\hat{O}(|\alpha|)} |\alpha\rangle_B \, \rho \right]$$

$$= \mathrm{Tr}\left[e^{it\hat{x}_\varphi} \rho \right]$$

This establishes pointwise convergence of characteristic function,
which in turn establishes pointwise convergence of probability distribution ~~for~~ homodyne detection.