Lecture 21

Holevo classification of single-mode Gaussian channels.

Given a single-mode input & output Gaussian quantum channel $\mathcal{N}$ characterized by 2×2 scaling matrix X & 2×2 noise matrix Y,

what are the possible ~~~~ classes of channels that can be realized up to arbitrary Gaussian unitaries acting on the input & output?

$$\mathcal{N}'(\rho) = U_{out}\, \mathcal{N}(U_{in}\, \rho\, U_{in}^{\dagger})\, U_{out}^{\dagger}$$

where U_{in} & U_{out} are Gaussian unitaries

1a

Let us call modified Gaussian channel $\mathcal{N}'$ w/ matrices X_c & Y_c for canonical

That is, in terms of covariance matrix transformations, we are considering

$$S_{out}\left(X\, S_{in}\, \sigma\, S_{in}^T\, X^T + Y\right) S_{out}^T$$

$$= S_{out}\, X\, S_{in}\, \sigma\, S_{in}^T\, X^T\, S_{out}^T + S_{out}\, Y\, S_{out}^T$$

Why is this classification possible? 1b

Situation simplifies b/c all 2x2 orthogonal matrices have form

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = R(\theta)\,\sigma_z$$

rotation w/ det $= 1$ (special orthogonal)

or

$$\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} = R(\theta)$$

rotation + reflection w/ det $= -1$

Also, all 2x2 real matrices X satisfy

$$X \Omega X^T = \det(X)\, \Omega$$

so then $\dfrac{X}{\sqrt{\det(X)}}$ is a symplectic matrix for this special case.
if $\det(X) > 0$

1c

$\forall \; \frac{X}{\sqrt{|\det(X)|}}$ is

anti symplectic when

$\det(X) < 0$

matrix M is antisymplectic

if $M \Omega M^T = -\Omega$

4 classes A, B, C, D

based on $\det(X)$

A) $\det(X) = 0$,

B) $\det(X) = 1$,

C) $\det(X) > 0 \wedge \det(X) \neq 1$,

D) $\det(X) < 0$. goto 2b

There are then subclasses to consider.

let us start w/

A) If $\det(X) = 0$, then either
$X = 0$ or X is rank one.

Suppose that $X = 0$

2b

canonical forms:

$A \to A_1$: $X_c = 0$, $Y_c = \nu I_2$ for $\nu \geq 1$

"replace w/ thermal state"

$A \to A_2$: $X_c = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $Y_c = \nu I_2$ for $\nu \geq 1$

$B \to B_1$: $X_c = I_2$, $Y_c = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

"phase-sensitive additive noise"

$B \to B_2$: $X_c = I_2$, $Y_c = \xi I_2$ w/ $\xi \geq 0$

"identity & additive-noise channels"

$C \to$ $X_c = \sqrt{\det(x)}\, I_2$, $Y_c = |\det(x) - 1|\nu I_2$ where $\nu \geq 1$

$\det(x) > 1$ "amplifier"

$0 < \det(x) < 1$ "thermal"

$D \to$ $X_c = \sqrt{|\det(x)|}\, \sigma_z$, $Y_c = (1 + |\det(x)|) \cdot \nu I_2$ where $\nu \geq 1$

"weak conjugate of amplifier"

→ back to 2

then due to the channel uncertainty relation

$$i X \Omega X^T + Y \geq i\Omega$$

if $X = 0$ then $Y \geq i\Omega$

then by Williamson,

$\exists$ symp. S_{out} such that

$$Y = S_{out} (\nu I_2) S_{out}^T$$

So then $Y_c = \nu I_2$ (i.e., apply S_{out}^{-1} to channel output)

& $X_c = 0$

<u>Interpretation</u>: channel is just trace & replace w/ a thermal state. of photon number $\bar{n} = \frac{\nu - 1}{2}$

$$N(\cdot) = \mathrm{Tr}[\cdot]\, \theta(\bar{n})$$

class is called A_1

If X is rank one,

~~then it has the form~~ $\cancel{\begin{bmatrix} x_1 & x_2 \\ cx_1 & cx_2 \end{bmatrix}}$ ~~for some reals~~ $\cancel{x_1, x_2, c}$

then $X i\Omega X^T = 0$ in this case because $\det(X) = 0$

& so we still have

$$Y \geq i\Omega$$

& symp. diagonalization of Y

as $S_{out} (\nu I_2) S_{out}^T = Y$

~~$S_{out} X$ has rank one~~

~~& thus has the form~~

$$\cancel{\begin{bmatrix} s_1 & c's_1 \\ s_2 & c's_2 \end{bmatrix}}$$

~~& when left multiplied by an arbitrary vector $[z_1, z_2]$~~

~~gives~~

So then channel is

$$X \sigma X^T + S_{out} (\nu I_2) S_{out}^T$$

Apply S_{out}^{-1} to output &

channel becomes

$$X' \sigma X'^T + \nu I_2$$

w/ $X' = S_{out}^{-1} X$

X' is rank one.

It has an SVD

$$X' = S_{out}' \begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix} S_{in}'$$

where S_{out}' & S_{in}' are

SO(2) & thus symp.

Apply $(S_{out}')^{-1}$ to output & $(S_{in}')^{-1}$ to input & channel becomes

$$\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix} \sigma \begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix} + \nu I_2$$

Now finally apply squeezer

$$\begin{bmatrix} x^{-1} & 0 \\ 0 & x \end{bmatrix}$$ to input

& channel becomes

$$X_c = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad Y_c = \nu I_2$$

$\boxed{\text{class } A_2}$

Now consider class B w/ $\det(X) = 1$

for <u>2x2 matrices</u> $\det(X) = 1$

actually implies that X is symplectic

$\Rightarrow$ X^{-1} is symp & so we

can apply X^{-1} @ channel

input to reduce channel to

$\sigma + Y$

Also since X is symplectic

$$\Rightarrow \quad iX\Lambda X^T + Y \geq i\Lambda$$

$$\Rightarrow \quad i\Lambda + Y \geq i\Lambda$$

$$\Rightarrow \quad Y \geq 0$$

sub cases: if $Y > 0$

then by Williamson theorem

$\exists$ symp S_{out} such that

$$Y = S_{out} (\nu I_2) S_{out}^T$$

where $\nu > 0$

then channel reduces to

(by applying S_{out}^{-1} to ch. output & S_{out} to ch. input)

$$X_c = I \qquad Y = \nu I_2$$

which is additive noise channel

if $Y = 0$, then we just have

$$X_c = I \qquad Y = 0$$

which is an identity channel

these form the class B_2

If Y is rank one, then

$\exists$ symplectic matr. S_{out} such that

$$Y = S_{out} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} S_{out}^T$$

then applying S_{out}^{-1} to ch. output + S_{out} to ch. input gives

$$X_c = I \quad , \quad Y = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

subclass B_1

$$\det(X) > 0 \qquad \det(X) \neq 1$$

for 2×2 matrices

$$S_1 = \frac{X}{\sqrt{\det(X)}} \quad \text{is a symplectic matrix}$$

then the condition

$$iX\Omega X^T + Y \geq i\Omega$$

$$\Rightarrow \quad Y \geq i\Omega - iX\Omega X^T$$

Supplement for class B_1

$Y \geq 0$ is rank one

$\Rightarrow$ ~~u~~usual eigendecomposition gives

$\lambda > 0$

$$Y = O \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix} O^T$$

where O has $\det(O) = 1$
& is thus symplectic

then $S' = \begin{bmatrix} \sqrt{\lambda} & 0 \\ 0 & \frac{1}{\sqrt{\lambda}} \end{bmatrix}$ is symp.

$= \begin{bmatrix} \sqrt{\lambda} & 0 \\ 0 & \sqrt{\frac{1}{\lambda}} \end{bmatrix}$

& so

$$Y = S \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} S^T$$

where $S = O S'$
is symplectic

$$X \cdot \Omega X^T = \det(X) \frac{X \Omega X^T}{\det(X)}$$
$$= \det(X)\, \Omega$$

$$\Rightarrow \quad Y \geq i\Omega\,(1 - \det(X))$$

take ~~the~~ transpose

$$Y \geq i\Omega\,(\det(X) - 1)$$
$$\Rightarrow \quad Y \geq i\Omega\,|\det(X) - 1|$$
$$\Rightarrow \quad \frac{Y}{|\det(X) - 1|} \geq i\Omega$$

So $\dfrac{Y}{|\det(X) - 1|}$ is a q. cov. mat.

$\Rightarrow \exists$ symp. matrix S_2 s.t.

$$\frac{Y}{|\det(X) - 1|} = S_2\,(\nu I_2)\,S_2^T$$

$$\Rightarrow \quad Y = |\det(X) - 1|\; S_2\, \nu I_2\, S_2^T$$

then applying $S_1^{-1} S_2$ at input

& S_2^{-1} at output gives

$$S_2^{-1} X S_1^{-1} S_2 = \sqrt{\det(X)}\, I_2$$

& $$S_2^{-1} Y S_2^{-T} =$$

$$|\det(X) - 1|\, \nu I_2$$

$$\Rightarrow \quad X_c = \sqrt{\det(X)}\, I_2$$

$$Y_c = |\det(X) - 1|\, \nu I_2$$

If $\det(X) \in (0,1)$, then

this is thermal channel

If $\det(X) > 1$, then

this is amplifier channel.

Class C.

so classes B_2 & C form additive noise, thermal & amp. channels

Next: class D

$\det(X) < 0$

for 2x2 matrices if $\det(X) < 0$,

then $\dfrac{X}{\sqrt{|\det(X)|}}$ is "antisymplectic"

meaning that

$$\frac{X \Omega X^T}{|\det(X)|} = -\Omega$$

sandwiching by σ_z gives

$$-\sigma_z \Omega \sigma_z^T = \Omega$$

$\Rightarrow$ $S_1 = \dfrac{\sigma_z X}{\sqrt{|\det(X)|}}$ is symplectic

then $iX\Omega X^T + Y \geq i\Omega$

$\Leftrightarrow$ $i\det(X)\Omega + Y \geq i\Omega$

$\Rightarrow$ $Y \geq i\Omega(1 - \det(X))$

$$Y \geq i\Omega\,(1 + |\det(X)|)$$

then $$\frac{Y}{1+|\det(X)|} \geq i\Omega$$

So $$\frac{Y}{1+|\det(X)|}$$ is a q. cov. matrix

$\Rightarrow \exists$ symplectic S s.t.

$$\frac{Y}{1+|\det(X)|} = S\,(\nu I_2)\,S^T$$

$$\Rightarrow Y = (1+|\det(X)|)\,S(\nu I_2)S^T$$

~~Apply~~ $\cancel{S_1^{-1}}$ ~~to channel input to get~~

$$\cancel{X S_1^{-1} = X X^{-1}\sigma_Z \sqrt{|\det(X)|}}$$

$$\cancel{= \sigma_Z \sqrt{|\det(X)|}}$$

Set $S_2 = X^{-1} S \sigma_z \sqrt{|\det(x)|}$

this is symplectic b/c

$\sqrt{|\det(x)|}\, X^{-1}$ & σ_z are

antisymplectic

Apply S_2 to channel input to get

$$X' = X S_2 = X X^{-1} S \sigma_z \sqrt{|\det(x)|}$$

$$= S \sigma_z \sqrt{|\det(x)|}$$

Now apply S^{-1} to channel output

$$S^{-1} X' = S^{-1} S \sigma_z \sqrt{|\det(x)|}$$

$$= \sigma_z \sqrt{|\det(x)|}$$

& $Y' = S^{-1} Y S^{-T}$

$$= \left(1 + |\det(x)|\right) \nu I_2$$

$$\Rightarrow X_c = \sigma_Z \sqrt{|\det(X)|}$$

$$Y_c = \nu I_2 \quad \text{for } \nu \geq 1$$

"weak conjugate of amplifier"