Lecture 20

Additive-noise channels

Suppose that a quantum Gaussian channel has $X = I$ (scaling matrix)

+ $Y > 0$ (noise matrix)

Recall that the action of the Gaussian channel adjoint on displacement operator is

$$\mathcal{N}^{+}(\hat{D}_{\Omega \underline{r}}) = \hat{D}_{\Omega X^T \underline{r}} \, e^{-\frac{1}{4}\underline{r}^T Y \underline{r}}$$

$\Rightarrow$ $\Omega \underline{r} \overset{\text{substitute}}{\to} \underline{r}'$ $\quad \Omega^T \underline{r}' = \underline{r}$

$$\Rightarrow \mathcal{N}^{+}(\hat{D}_{\underline{r}'}) = \cancel{\hat{D}_{\Omega X^T \Omega \underline{r}} \, e^{-\frac{1}{4}\underline{r}^T \Omega^T Y \Omega \underline{r}}}$$

$$= \hat{D}_{\Omega X^T \Omega^T \underline{r}'} \, e^{-\frac{1}{4}\underline{r}'^T \Omega Y \Omega^T \underline{r}'}$$

Then characteristic function of output in case of additive-noise channel is

$$\chi_{N(\rho)}(\underline{r}) = \mathrm{Tr}\{\hat{D}_{-\underline{r}}\, N(\rho)\}$$

$$= \mathrm{Tr}\left[N^{\dagger}(\hat{D}_{-\underline{r}})\,\rho\right]$$

$$= \mathrm{Tr}\left[\hat{D}_{\Omega X^{T}\Omega^{T}(-\underline{r})}\,\rho\right] e^{-\frac{1}{4} r^{T}\Omega Y \Omega^{T}\underline{r}}$$

$X = I$

$$\Rightarrow \quad = \mathrm{Tr}\left[\hat{D}_{-\underline{r}}\,\rho\right] e^{-\frac{1}{4} r^{T}\Omega Y \Omega^{T}\underline{r}}$$

$$= \chi_{\rho}(\underline{r})\, e^{-\frac{1}{4} r^{T}\Omega Y \Omega^{T}\underline{r}}$$

Then ~~Since~~ $N(\rho)$ is given by

$$N(\rho) = \frac{1}{(2\pi)^{n}} \int_{\mathbb{R}^{2n}} d\underline{r}\; \chi_{N(\rho)}(\underline{r})\, \hat{D}_{\underline{r}}$$

$$= \frac{1}{(2\pi)^{n}} \int_{\mathbb{R}^{2n}} d\underline{r}\; \chi_{\rho}(\underline{r})\, \hat{D}_{\underline{r}}\, e^{-\frac{1}{4} r^{T}\Omega Y \Omega^{T}\underline{r}}$$

Now consider that

$$(*) = \int_{\mathbb{R}^{2n}} d\underline{r}' \, \frac{e^{-\underline{r}'^T Y^{-1} \underline{r}'}}{\pi^n \sqrt{\mathrm{Det}(Y)}} \, \hat{D}_{\underline{r}'} \, \rho \, \hat{D}_{\underline{r}'}^{+}$$

$$= \iint_{\mathbb{R}^{2n}} d\underline{r}' \, d\underline{r} \, \frac{e^{-\underline{r}'^T Y^{-1} \underline{r}'}}{\pi^n \sqrt{\mathrm{Det}(Y)}} \, \chi_\rho(\underline{r}) \, \hat{D}_{\underline{r}'} \hat{D}_{\underline{r}} \hat{D}_{\underline{r}'}^{+}$$

$$= \iint d\underline{r}' \, d\underline{r} \quad '' \quad \chi_\rho(\underline{r}) \, \hat{D}_{\underline{r}} \, e^{-i \underbrace{\underline{r}^{\prime T}}_{(\underline{r}')^T} \Omega \underline{r}}$$

$$= \int d\underline{r} \; \chi_\rho(\underline{r}) \, \hat{D}_{\underline{r}} \underbrace{\int d\underline{r}' \, \frac{e^{-\underline{r}'^T Y^{-1} \underline{r}'}}{\pi^n \sqrt{\mathrm{Det}(Y)}} \, e^{-i \underline{r}'^T \Omega \underline{r}}}_{\text{Fourier transform of multivariate Gaussian (using Gaussian integration formula)}}$$

$$\int d\underline{r}' \; e^{-\underline{r}'^T A \underline{r}' + \underline{r}'^T b}$$

$$= \frac{\pi^n}{\sqrt{Det(A)}} e^{\frac{1}{4} b^T A^{-1} b}$$

w/ $A = Y^{-1}$

$b = -i\Omega \underline{r}$

integral w/o normalization is

$$\Rightarrow \quad = \frac{\pi^n}{\sqrt{Det(Y^{-1})}} e^{-\frac{1}{4} r^T \Omega Y \Omega^T \underline{r}}$$

$$\Rightarrow \quad (*) = \int d\underline{r}\, \chi_\rho(\underline{r})\, \hat{D}_r\, e^{-\frac{1}{4}\underline{r}^T \Omega Y \Omega^T \underline{r}}$$

So all of this means that an additive noise channel w/ $X = I$ & $Y > 0$ can be written as

$$\mathcal{N}(\rho) = \int_{\mathbb{R}^{2n}} d\underline{r}' \frac{e^{-r'^T Y^{-1} r'}}{\pi^n \sqrt{Det(Y)}} \hat{D}_{\underline{r}'}\, \rho\, \hat{D}_{\underline{r}'}^\dagger$$

i.e., Gaussian mixture of random displacements.

this is an important physical interpretation useful in a variety of contexts.

Interesting consequences:

Any thermal state $\theta(\bar{n})$ is the result of sending a vacuum state through a single-mode additive-noise channel:

$$\theta(\bar{n}) = \frac{1}{\pi \bar{n}} \int d^2\alpha \; e^{-\frac{|\alpha|^2}{\bar{n}}} \hat{D}_\alpha |0\rangle\langle 0| \hat{D}_{-\alpha}$$

Any n-mode Gaussian state is the result of transmitting a pure Gaussian state through an additive-noise channel.

To see this, let σ be the cov. matrix for an arbitrary Gaussian state ρ.

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By Williamson theorem, it can be written as

$\sigma = S \underline{\underline{\nu}} S^T$ where

S is symplectic &

$$\underline{\underline{\nu}} = \bigoplus_{j=1}^{n} \nu_j I_2$$

where $\nu_j \geq 1$ are symplectic eigenvalues.

Recall that a Gaussian state is pure if and only if $\nu = I$

& so the Gaussian state ρ_P w/ covariance matrix $\sigma_P = SS^T$ is pure.

Then the Gaussian additive noise channel $\mathcal{N}$ w/ $X = I$ & $Y = \sigma - \sigma_P \geq 0$ is such that $\mathcal{N}(\rho_P)$ has mean vector equal to the original & covariance matrix given by

$$X \sigma_p X^T + Y$$

$$= \sigma_p + \sigma - \sigma_p$$

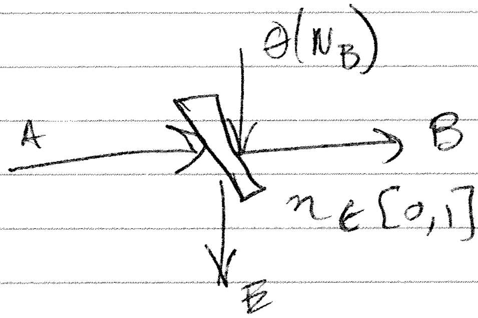
$$= \sigma .$$

So this means that any Gaussian state w/ covariance matrix $= \sigma$ can be written as

$$\rho = \int_{\mathbb{R}^{2n}} d\underline{r} \; \frac{e^{-\underline{r}^T (\sigma - \sigma_p)^{-1} \underline{r}}}{\pi^n \sqrt{\mathrm{Det}(\sigma - \sigma_p)}} \hat{D}_r \rho_p \hat{D}_r^\dagger$$

(assuming $\sigma > \sigma_p$)

Important for understanding Gaussian entanglement of formation thermal channel is characterized by beamsplitter interaction of channel input w/ thermal environment state, followed by partial trace over environment.



$\mathcal{L}_{\eta, N_B}$

Thermal channel has

$$X = \sqrt{\eta}\, I_2 \quad , \quad Y = (2N_B + 1)(1-\eta)\, I_2$$

w/ $\eta \in [0,1]$ & $N_B \geq 0$.

Pure-loss channel $\mathcal{L}_\eta \equiv \mathcal{L}_{\eta,0}$ when $N_B = 0$

can also adopt parametrization

$\sqrt{\eta} = \cos\theta \qquad \sqrt{1-\eta} = \sin\theta$

Proof that thermal channel results from beam-splitter interaction

$$\begin{bmatrix} \sqrt{\eta}\, I_2 & \sqrt{1-\eta}\, I_2 \\ -\sqrt{1-\eta}\, I_2 & \sqrt{\eta}\, I_2 \end{bmatrix} = B(\eta)$$

then on single-mode input w/ covariance matrix σ, we find that

$$B(\eta) \begin{bmatrix} \sigma & 0 \\ 0 & (2N_B+1) I_2 \end{bmatrix} B^T(\eta)$$

$$= \begin{bmatrix} \eta\sigma + (1-\eta)(2N_B+1) I_2 & \cdot\cdot \\ \cdot\cdot & \cdot\cdot \end{bmatrix}$$

upper left block is output of channel after partial trace.

can use characteristic functions to argue that if the output of a pure-loss channel is a pure state, then it follows that the input is a coherent state (necessary condition).

Amplifier channels

Amplifier channel has $\mathcal{A}_{G,N_B}$

$$X = \sqrt{G}\, I_2 \quad \& \quad Y = (G-1)(2N_B+1) I_2$$

for $G \geq 1$ & $N_B \geq 0$

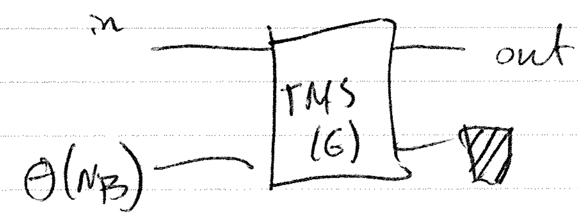
Alternate parametrization

$$\sqrt{G} = \cosh r \quad \& \quad G-1 = (\sinh r)^2$$

for $r \geq 0$

When $N_B = 0$, known as "pure amplifier" or "quantum-limited amplifier" pure amplifier $\mathcal{A}_G \equiv \mathcal{A}_{G,0}$

the amplifier can be physically realized in terms of a two-mode squeezer acting on channel input & thermal environment state, followed by partial trace



$$S(G) = \begin{bmatrix} \sqrt{G}\, I_2 & \sqrt{G-1}\, \sigma_Z \\ \sqrt{G-1}\, \sigma_Z & \sqrt{G}\, I_2 \end{bmatrix}$$

then

$$S(G) \begin{bmatrix} \sigma & \\ & (2N_B+1) I_2 \end{bmatrix} S^T(G)$$

$$= \begin{bmatrix} G\sigma + (G-1)(2N_B+1) I_2 & \cdots \\ \cdots & \cdots \end{bmatrix}$$

upper left block is channel output

Channel Decompositions:

We can combine pure-loss channels w/ pure amplifier channels to realize additive-noise channels:

$$\mathcal{A}_{1/\eta} \circ \mathcal{L}_\eta = \mathcal{T}_{2(\frac{1}{\eta}-1)}$$

Proof: (simple)

consider action on covariance matrices

$$\sigma \xrightarrow{\text{pure-loss}} \eta\sigma + (1-\eta) I_2$$

$$\xrightarrow{\text{pure-amplifier}} G\left(\eta\sigma + (1-\eta) I_2\right) + (G-1) I_2$$

$$= G\eta\sigma + \left\{G(1-\eta) + (G-1)\right\} I_2$$

taking $G = 1/\eta$ gives

$$\sigma \rightarrow \sigma + 2\left(\tfrac{1}{\eta}-1\right) I_2$$

Phase-insensitive Gaussian channels

A ~~Gaussian~~ channel $\mathcal{N}$ is phase-insensitive or phase covariant if

$$(*) \quad \mathcal{N}(e^{-i\hat{n}\phi}\rho e^{i\hat{n}\phi}) = e^{-i\hat{n}\phi}\mathcal{N}(\rho)e^{i\hat{n}\phi}$$

$\forall$ input states ρ & phases $\phi \in \mathbb{R}$

If a channel is not phase insensitive, it can be made phase-insensitive by the following phase twirling procedure:

$$\frac{1}{2\pi}\int_0^{2\pi} d\phi \; e^{i\hat{n}\phi}\,\mathcal{N}(e^{-i\hat{n}\phi}\rho e^{i\hat{n}\phi})\,e^{-i\hat{n}\phi}$$

If a single-mode Gaussian channel is phase-insensitive, its form is restricted.

We can prove this by using the Choi state

Suppose that the covariance condition in (*) holds

then consider that

$$\frac{1}{2\pi}\int_0^{2\pi} d\phi \; e^{i\hat{n}_B\phi}\, \mathcal{N}_{A\to B}\left(e^{-i\hat{n}_A\phi}\, \psi(\bar{n})_{RA}\, e^{i\hat{n}_A\phi}\right) e^{-i\hat{n}_B\phi}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} d\phi \left(e^{-i\hat{n}_R\phi} \otimes e^{i\hat{n}_B\phi}\right) \mathcal{N}_{A\to B}\left(\psi(\bar{n})_{RA}\right) \left(e^{-i\hat{n}_R\phi} \otimes e^{i\hat{n}_B\phi}\right)^\dagger$$

$$= \mathcal{N}_{A\to B}\left(\psi(\bar{n})_{RA}\right) \qquad \text{(by covariance condition)}$$

where we used that

$$e^{-i\hat{n}_A\phi}\, |\psi(\bar{n})\rangle_{RA}$$

$$= e^{-i\hat{n}_R\phi}\, |\psi(\bar{n})\rangle_{RA}$$

Let σ_{RB} be covariance matrix for $\mathcal{N}_{A\to B}(\psi(\bar{n})_{RA})$

Then covariance matrix for twirled channel $\bar{\mathcal{B}}$

$$\frac{1}{2\pi}\int_0^{2\pi} d\phi \; X(\phi)\, \sigma_{RB}\, X^T(\phi) = \bar{\sigma}_{RB}$$

where $X(\phi) = \begin{bmatrix} c(\phi) & s(\phi) & 0 & 0 \\ -s(\phi) & c(\phi) & 0 & 0 \\ 0 & 0 & c(\phi) & -s(\phi) \\ 0 & 0 & s(\phi) & c(\phi) \end{bmatrix}$

But $\bar{\sigma}_{RB}$ has the form

$$\begin{bmatrix} a I_2 & z R(\theta) \\ z R(\theta) & b I_2 \end{bmatrix}$$

where $a = 2\bar{n}+1$

$z \geq 0 \qquad \theta \in [0, 2\pi] \qquad b \geq 1$

Since Choi ~~for~~ Gaussian must have form

$$\begin{bmatrix} (2\bar{n}+1) I_2 & 2\sqrt{\bar{n}(\bar{n}+1)}\,\sigma_z X^T \\ 2\sqrt{\bar{n}(\bar{n}+1)}\,X\sigma_z & (2\bar{n}+1)XX^T+Y \end{bmatrix}$$

we can infer that

$$X = \frac{z}{2\sqrt{\bar{n}(\bar{n}+1)}}\begin{bmatrix} c(\theta) & -s(\theta) \\ s(\theta) & c(\theta) \end{bmatrix}$$

$$Y = bI_2 - (2\bar{n}+1)XX^T$$

$$= \left[b - \frac{(2\bar{n}+1)z^2}{4\bar{n}(\bar{n}+1)}\right] I_2$$

One can then apply a unitary pre-processing to channel to get $X = x I_2$ where

$Y = y I_2$ $\quad y \geq |x^2 - 1|$

(CPTP condition)

$iX\Omega X^T + Y \geq i\Omega$