Lecture 19

Stinespring dilation theorem for Gaussian channels:

- Any Gaussian channel can be realized by the action of a Gaussian unitary on the input state & an environment prepared in a Gaussian state
- Also, any Gaussian unitary that interacts the system w/ a Gaussian environment state leads to a Gaussian channel w/ X & Y satisfying

$$iX\Omega_n X^T + Y \geq i\Omega_m$$

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Note that this condition implies that $Y \geq 0$.

let ψ be a real $2m \times 1$ vector. Then

$$iX\Lambda_n X^T + Y - i\Lambda_m \geq 0$$

$$\Rightarrow \psi^T(iX\Lambda_n X^T + Y - i\Lambda_m)\psi \geq 0$$

$$\Rightarrow i(X^T\psi)^T\Lambda_n(X^T\psi) + \psi^T Y \psi - i\psi^T\Lambda_m\psi \geq 0$$

$$\Rightarrow \psi^T Y \psi \geq 0 \quad \forall \psi$$

$$\Rightarrow Y \geq 0.$$

Let us establish the 2nd statement first.

Let S be a symplectic matrix such that

$$S = \begin{array}{c} \begin{array}{cc} 2n & 2l \end{array} \\ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \end{array} \begin{array}{l} \}\, 2m \\ \}\, 2(n+l-m) \end{array}$$

where n is # of input modes,

m is # of output modes,

& l is # of environment modes.

Since S is symplectic, $A, B, C,$ & D satisfy

$$\Omega_{n+l} = S\, \Omega_{n+l}\, S^T$$

$$= \left[\begin{array}{c|c} A \Omega_n A^T + B \Omega_l B^T & A \Omega_n C^T + B \Omega_l D^T \\ \hline C \Omega_n A^T + D \Omega_l B^T & C \Omega_n C^T + D \Omega_l D^T \end{array}\right]$$

$$= \begin{bmatrix} \Omega_m & 0 \\ 0 & \Omega_{n+l-m} \end{bmatrix}$$

We get the action of the channel by tracing out the $n+l-m$ output environment modes, i.e., by considering the upper left $2m \times 2m$ block of

$$S(\sigma \oplus \sigma_E) S^T$$

where σ_E is the covariance matrix for the initial state of the environment.

The matrix σ_E is thus constrained to satisfy $\sigma_E + i\Omega_l \geq 0$ $(\Leftrightarrow \sigma_E \geq i\Omega_l)$

$\Rightarrow$ output state cov. matrix is given by $A\sigma A^T + B\sigma_E B^T$

Then $\sigma_E + i\Lambda_e \geq 0$

$\Rightarrow$ $B\sigma_E B^T + iB\Lambda_e B^T \geq 0$

Using the constraint

$$B\Lambda_e B^T = \Lambda_m - A\Lambda_n A^T$$

$\Rightarrow$ $B\sigma_E B^T + i\Lambda_m - iA\Lambda_n A^T \geq 0$

Identify $X = A$ &
$Y = B\sigma_E B^T$ &
connect w/ previous description of a Gaussian channel.

then this implements a Gaussian channel according to the previous definition.

Now let us prove the other direction:

Given $2m \times 2n$ real matrix X
& $2m \times 2m$ real symmetric Y
satisfying

$$i X \Omega_n X^T + Y \geq i \Omega_m$$

$\exists$ a Gaussian unitary U
s.t.

$$\mathcal{N}(\rho) = \mathrm{Tr}_E \left[U \left(\rho \otimes |0\rangle\langle 0| \right) U^\dagger \right]$$

where this is a tensor-power vacuum state

For covariance matrices, evolution is given by

$$\sigma \to X \sigma X^T + Y$$

We want to find symplectic matrix $S = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$

such that $A = X$

Initial state of environment will be tensor-power vacuum w/ C.M. $\sigma_E = I$. ($2\ell \times 2\ell$ identity) w/ ℓ

Recalling that $Y \geq 0$, Can pick $B = \sqrt{Y}\, O$

for some ~~orthogonal~~ matrix O, w/ orthonormal rows ($2m \times 2\ell$ matrix)

Set $\ell = 2m$

Then reduced evolution is

$$A \sigma A^T + B \sigma_E B^T$$

$$= X \sigma X^T + \sqrt{Y}\, O\, I\, O^T \sqrt{Y}$$

$$= X \sigma X^T + Y$$

So we need to figure out how to pick C + D such that S is symplectic.

Previously, we showed that symplectic condition $S \Omega S^T = \Omega$ is equivalent to

$$A \Omega_n A^T + B \Omega_\ell B^T = \Omega_m, \quad (*)$$

$$A \Omega_n C^T + B \Omega_\ell D^T = 0,$$

$$C \Omega_n A^T + D \Omega_\ell B^T = 0,$$

$$C \Omega_n C^T + D \Omega_\ell D^T = \Omega_{n+\ell-m}$$

Then matrix O should be chosen such that $(*)$ is satisfied, i.e.,

$$X \Omega_n X^T + \sqrt{Y} O \Omega_\ell O^T \sqrt{Y} = \Omega_m$$

For simplicity, consider the case of pos. definite $Y > 0$.

Then rewrite condition as

$$O \Lambda_e O^T = \underbrace{Y^{-1/2}(\Lambda_m - X \Lambda_n X^T) Y^{-1/2}}_{\text{this matrix is antisymmetric}}$$

It thus can be brought into canonical form by orthogonal $2m \times 2m$ matrix R

$$R Y^{-1/2} (\Lambda_m - X \Lambda_n X^T) Y^{-1/2} R^T$$

$$= \bigoplus_{j=1}^{m} d_j \Lambda_1$$

the condition

$$Y + i\Lambda_m \geq i X \Lambda_n X^T \iff$$

$$Y \geq i\Lambda_m - iX\Lambda_n X^T$$ (by moving around & transpose)

$$\Leftrightarrow \quad I \geq Y^{-1/2}(i\Lambda_m - iX\Lambda_n X^T)Y^{-1/2}$$

$$\Rightarrow \quad d_j \leq 1 \quad \forall j$$

Now consider that by setting

$$O_j = \begin{pmatrix} \cos\theta_j & 0 & -\sin\theta_j & 0 \\ 0 & \cos\theta_j & 0 & \sin\theta_j \end{pmatrix}$$

such that $\cos(2\theta_j) = d_j$,

we find that

$$O_j O_j^T = I_2 \qquad O_j \Lambda_2 O_j^T = \begin{bmatrix} 0 & d_j \\ -d_j & 0 \end{bmatrix}$$

$\Rightarrow$ $M = \bigoplus_{j=1}^{m} O_j$ is s.t.

$$M \Lambda_\ell M^T = R\, Y^{-1/2}(\Lambda_m - X\Lambda_n X^T)Y^{-1/2} R^T$$

Note that M has form

$$M = \begin{bmatrix} O_1 & \underline{o} & \underline{o} & \underline{o} \\ \underline{o} & O_2 & & \\ \underline{o} & & \ddots & \\ \vdots & & & \\ \underline{o} & & & O_m \end{bmatrix}$$

then set $O = R^T M$ &

$$\Rightarrow \quad O \Lambda_e O^T = Y^{-1/2}(\Lambda_m - X \Lambda_n X^T) Y^{-1/2}$$

So then we have A & B

s.t. $A = X$ & $B = \sqrt{Y}\, O$ ($\Rightarrow BB^T = Y$)

& $A \Lambda_n A^T + B \Lambda_e B^T = \Lambda_m$

- can restate the latter condition as saying that the rows v_j the matrix $[A \ B]$

satisfy $\quad v_j \Lambda_{m+(n+m)} v_k^T = \Lambda_{jk}$

where $\Lambda_{m+(n+m)} = \begin{bmatrix} \Lambda_m & 0 \\ 0 & \Lambda_{n+m} \end{bmatrix}$

The remainder of the proof is regarding ~~extending~~ finding matrices C & D such that

$$S = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$ is a

symplectic matrix.

- This is equivalent to extending the symplectic basis formed by the rows of $[A \; B]$ to a complete symplectic basis for the $m+n+m$ space. Already have m linearly independent vectors

- Procedure consists of finding $n+m$ other linearly independent vectors & performing a symplectic Gram-Schmidt orthogonalization

Some special cases of channels:

1) unitary evolution + noise

Suppose that X is symplectic so that $n=m$.

Then condition

$$iX\Lambda_n X^T + Y \geq i\Lambda_m$$

becomes

$$i\Lambda_n + Y \geq i\Lambda_n$$

$\Rightarrow$ $Y \geq 0$ & channel can be thought of as the composition of unitary evolution according to symp. matrix X followed by additive noise acc'ding to Y.

Action of the adjoint of a Gaussian channel on displacement operators:

Let $\mathcal{N}$ be a Gaussian channel characterized by X & Y. Then

$$\mathcal{N}^{\dagger}(\hat{D}_{\Omega \underline{r}}) = \hat{D}_{\Omega X^T \underline{r}}\, e^{-\frac{1}{4}\underline{r}^T Y \underline{r}}$$

Proof: For arbitrary operator $\hat{A}$, Gaussian Stinespring dilation allows for writing

$$\mathcal{N}(\hat{A}) = \mathrm{Tr}_E\left[\hat{S}^{\dagger}(\hat{A} \otimes |0\rangle\langle 0|)\hat{S}\right]$$

for Gaussian unitary $\hat{S}$ w/ symp. transformation S satisfying

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$$\hat{S}\hat{\underset{\approx}{R}}\hat{S}^{\dagger} = S\hat{\underset{\approx}{R}}$$

($\hat{\underset{\approx}{R}}$ is vector of canonical op's for bipartite system)

$$\hat{S}\hat{D}_R\hat{S}^{\dagger} = \hat{D}_{S^{-1}R}$$

+ R is real vector.

S can be written in block form as

$$S = \begin{pmatrix} X & \sqrt{Y}\,O \\ * & * \end{pmatrix}$$

Then, from the definition of adjoint, it follows that

$$\mathcal{N}^{\dagger}(\hat{B}) = \mathrm{Tr}_E\left[\left(I \otimes |0\rangle\langle 0|\right)\hat{S}(\hat{B}\otimes\hat{I})\hat{S}^{\dagger}\right]$$

then set $\hat{B} = \hat{D}_{\Lambda r}$ to find that

$$\mathcal{N}^{+}(\hat{D}_{\Lambda \underline{r}})$$

$$= \mathrm{Tr}_E\left[(I \otimes |0\rangle\langle 0|)\ \hat{S}\ (\hat{D}_{\Lambda \underline{r}} \otimes \hat{I})\ \hat{S}^{+} \right]$$

$$= \mathrm{Tr}_E\left[(I \otimes |0\rangle\langle 0|)\ \hat{S}\ \hat{D}_{\Lambda(\underline{r} \oplus 0)}\ \hat{S}^{+} \right]$$

$$= \mathrm{Tr}_E\left[(I \otimes |0\rangle\langle 0|)\ \hat{D}_{S^{-1}\Lambda(\underline{r} \oplus 0)} \right]$$

$$= \mathrm{Tr}_E\left[(I \otimes |0\rangle\langle 0|)\ \hat{D}_{\Lambda S^{T}(\underline{r} \oplus 0)} \right]$$

$$= \mathrm{Tr}_E\left[(I \otimes |0\rangle\langle 0|)\ \hat{D}_{\Lambda X^{T}\underline{r} \oplus \Lambda O^{T}\sqrt{Y}\underline{r}} \right]$$

$$= \hat{D}_{\Lambda X^{T}\underline{r}}\ \mathrm{Tr}\left[|0\rangle\langle 0|\ \hat{D}_{\Lambda O^{T}\sqrt{Y}\underline{r}} \right]$$

$$= \hat{D}_{\Lambda X^{T}\underline{r}}\ \chi_{|0\rangle\langle 0|}(\Lambda O^{T}\sqrt{Y}\underline{r})$$

Now using that $\chi_{|0\rangle\langle 0|}(\underline{r}) = e^{-\frac{1}{4}|\underline{r}|^2}$

$$= \hat{D}_{\Lambda X^{T}\underline{r}}\ e^{-\frac{1}{4}|\Lambda O^{T}\sqrt{Y}\underline{r}|^2}$$

$$= \hat{D}_{\Lambda X^{T}\underline{r}}\ e^{-\frac{1}{4}\underline{r}^{T} Y \underline{r}}$$