Lecture 18

An important state in Gaussian QI is the two-mode squeezed vacuum state.

It is generated by the action of a two-mode squeezing transformation on a two-mode vacuum state:

$$\hat{S}_r |0\rangle|0\rangle$$

where

$$\hat{S}_r = e^{-\frac{i}{2}\hat{r}^T(rH_{TMS})\hat{r}}$$

where $H_{TMS} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ & $r \geq 0$ is squeezing strength

so that

$$\hat{H}_{TMS} = i r (\hat{a}_1^\dagger \hat{a}_2^\dagger - \hat{a}_1 \hat{a}_2) \quad \& \quad \hat{S}_r = e^{-i\hat{H}_{TMS}}$$

& symplectic transformation is

$$S_r = \begin{bmatrix} \cosh r \, I_2 & \sinh r \, \sigma_z \\ \sinh r \, \sigma_z & \cosh r \, I_2 \end{bmatrix}$$

$$S_r = \begin{bmatrix} \cosh r & 0 & \sinh r & 0 \\ 0 & \cosh r & 0 & -\sinh r \\ \sinh r & 0 & \cosh r & 0 \\ 0 & -\sinh r & 0 & \cosh r \end{bmatrix}$$

Since the covariance matrix for the two-mode vacuum state is I_4, this implies that the covariance matrix for the two-mode squeezed vacuum state is

$$S_r I_4 S_r^T = S_r S_r^T$$

$\cosh 2r$ $\qquad\qquad$ $\sinh 2r$

$$= \begin{bmatrix} \cosh^2 r \, I_2 & \sinh^2 r \, \sigma_z \\ \sinh^2 r \, \sigma_z & \cosh^2 r \, I_2 \end{bmatrix}$$

- Mean vector $= 0$ since original mean vec $= 0$.
- Notice that the covariance matrix for the reduced density operator is $\cosh r \, I_2$

Given that $\cosh r \geq 1 \;\forall\, r \in \mathbb{R}$

& cov. matrix B $\propto$ I_2

this is the ~~covarjarice~~ covariance matrix for a thermal state.

We can set $\cancel{\cosh r}\ \cosh(2r) = 2\bar{n}+1$

& this implies that $\sinh r = 2\sqrt{\bar{n}(\bar{n}+1)}$

$\Rightarrow$ cov. matrix for TMSV is

$$\begin{bmatrix} (2\bar{n}+1)\, I_2 & 2\sqrt{\bar{n}(\bar{n}+1)}\,\sigma_z \\ 2\sqrt{\bar{n}(\bar{n}+1)}\,\sigma_z & (2\bar{n}+1)\, I_2 \end{bmatrix}$$

In Hilbert space, the state is given by

$$|\psi_r\rangle = e^{-i\hat{H}_{TMS}}|0\rangle|0\rangle = e^{r(\hat{a}_1^\dagger \hat{a}_2^\dagger - \hat{a}_1\hat{a}_2)}|0\rangle|0\rangle$$

can find this by means of disentangling theorem, itself a consequence of BCH:

$$e^{r(\hat{a}_1^\dagger \hat{a}_2^\dagger - \hat{a}_1 \hat{a}_2)}$$

$$= e^{\Gamma \hat{a}_1^\dagger \hat{a}_2^\dagger} \, e^{-g(\hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2 + 1)} \, e^{-\Gamma \hat{a}_1 \hat{a}_2}$$

where $\Gamma = \tanh(r)$

$g = \ln(\cosh(r))$

Consider that

$$e^{-\Gamma \hat{a}_1 \hat{a}_2} |0\rangle|0\rangle$$

$$= \sum_{l=0}^{\infty} \frac{(-\Gamma \hat{a}_1 \hat{a}_2)^l}{l!} |0\rangle|0\rangle = |0\rangle|0\rangle$$

then $e^{-g(\hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2 + 1)} |0\rangle|0\rangle$

$$= e^{-g(\hat{a}_1^\dagger \hat{a}_1)} \, e^{-g(\hat{a}_2^\dagger \hat{a}_2)} \, e^{-g} |0\rangle|0\rangle$$

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$$= e^{-g} |0\rangle|0\rangle \qquad \left(\text{since } e^{-g\,\hat{a}_i^\dagger \hat{a}_i}|0\rangle = e^{-g\cdot 0}|0\rangle = |0\rangle\right)$$

$$= \frac{1}{\cosh r} |0\rangle|0\rangle$$

Finally,

$$e^{\Gamma \hat{a}_1^\dagger \hat{a}_2^\dagger} |0\rangle|0\rangle$$

$$= \sum_{l=0}^{\infty} \frac{\left(\Gamma \hat{a}_1^\dagger \hat{a}_2^\dagger\right)^l}{l!} |0\rangle|0\rangle$$

$$= \sum_{l=0}^{\infty} (\tanh r)^l \frac{\hat{a}_1^\dagger}{\sqrt{l!}} \frac{\hat{a}_2^\dagger}{\sqrt{l!}} |0\rangle|0\rangle$$

$$= \sum_{l=0}^{\infty} (\tanh r)^l |l\rangle|l\rangle$$

$$\Rightarrow |\Psi_r\rangle = \frac{1}{\cosh r} \sum_{n=0}^{\infty} (\tanh r)^n |n\rangle|n\rangle$$

For the other parametrization, we have

$$\frac{1}{\cosh r} = \frac{1}{\sqrt{\bar{n}+1}}$$

& $\tanh r = \sqrt{\frac{\bar{n}}{\bar{n}+1}}$

$$\Rightarrow |\psi_r\rangle = \frac{1}{\sqrt{\bar{n}+1}} \sum_{n=0}^{\infty} \left(\sqrt{\frac{\bar{n}}{\bar{n}+1}}\right)^n |n\rangle|n\rangle$$

Gaussian quantum channels

previously, we discussed Gaussian unitaries. Now we consider channels that map Gaussian states to Gaussian states.

- Gaussian channels maps an n-mode state to an m-mode state is characterized by

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by $2m \times 2n$ real matrix X,

called <u>scaling matrix</u>, &

$2m \times 2m$ real symmetric matrix

Y, called <u>noise matrix</u>,

& a displacement δ where

$\delta \in \mathbb{R}^{2m}$.

The effect of the channel on

an input Gaussian state w/

mean vector $\bar{r} \in \mathbb{R}^{2n}$ & cov.

matrix $\sigma \in \mathbb{R}^{2n \times 2n}$ is

$$\bar{r} \to X\bar{r} + \delta$$

$$\sigma \to X \sigma X^T + Y$$

X & Y should satisfy

$$Y + i\Omega_m \geq iX\Omega_n X^T$$

This condition comes about in order for the output state of the channel to be a legitimate state (respecting the uncertainty relation $\sigma + i\Omega \geq 0$)

We can prove one direction:

If X & Y correspond to a Gaussian channel, then $Y + i\Omega \geq iX\Omega X^T$.

Consider the action of the channel $\mathcal{N}$ on $|\psi_{r,n}\rangle_{RA} = |\psi_r\rangle^{\otimes n}$

(A is n-mode system)

$$(\mathrm{id}_R \otimes \mathcal{N}_{A \to B})(|\psi_{r,n}\rangle\langle\psi_{r,n}|_{RA})$$

Consider that cov. matrix of $|\psi_{r,n}\rangle_{RA}$ is

$$\sigma_n(r) = \begin{bmatrix} \cosh(2r)\, I_{2n} & \sinh(2r)\, \Sigma_n \\ \sinh(2r)\, \Sigma_n & \cosh(2r)\, I_{2n} \end{bmatrix}$$

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where $\Sigma_n = \bigoplus_{j=1}^{n} \sigma_z = I_n \otimes \sigma_z$

Then covariance matrix of output state is

$$\begin{bmatrix} I_{2n} & 0 \\ 0 & X \end{bmatrix} \begin{bmatrix} \sigma_n(r) \end{bmatrix} \begin{bmatrix} I_{2n} & 0 \\ 0 & X \end{bmatrix} + \begin{bmatrix} 0_{2n} & 0 \\ 0 & Y \end{bmatrix}$$

$$= \begin{bmatrix} \cosh(2r) I_{2n} & \sinh(2r) \Sigma_n X^T \\ \sinh(2r) X \Sigma_n & \cosh(2r) X X^T + Y \end{bmatrix}$$

$= \sigma_{out}(r)$

Given that $\mathcal{N}$ is a channel,

this is a legitimate covariance matrix, satisfying

$$\sigma_{out}(r) + i \Lambda_{n+m} \geq 0$$

or

$$\begin{bmatrix} \cosh(2r) I_{2n} + i \Lambda_n & \sinh(2r) \Sigma_n X^T \\ \sinh(2r) X \Sigma_n & \cosh(2r) X X^T + Y + i \Lambda_m \end{bmatrix} \geq 0$$

Now invoke Schur complement lemma:

$$Z = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} > 0 \iff$$

$$A > 0 \quad \& \quad C - B^T A^{-1} B > 0$$

(holds for pos. def., but can be generalized to PSD case)

So previous condition is equivalent to

$$\cosh(2r)\, I_{2n} + i\Omega_n > 0 \quad \&$$

(*) $$\cosh(2r)\, X X^T + Y + i\Omega_m$$

$$- \sinh(2r)\, X \Sigma_n \left(\cosh(2r)\, I_{2n} + i\Omega_n \right)^{-1}$$

$$\sinh(2r)\, \Sigma_n X^T > 0$$

Consider that

$$\begin{bmatrix} \cosh(2r) I_2 \\ + i\Omega_1 \end{bmatrix}^{-1} = \begin{bmatrix} \cosh(2r) & i \\ -i & \cosh(2r) \end{bmatrix}^{-1}$$

$$= \frac{\cosh(2r) I_2 - i\Omega_1}{\sinh^2(2r)}$$

$$\Rightarrow (*) = \cosh(2r) X X^T + Y + i\Omega_m - \sinh^2(2r)\, X \Sigma_n \left(\frac{\cosh(2r) I_{2n} - i\Omega_n}{\sinh^2(2r)} \right) \Sigma_n X^T$$

$$= \cosh(2r) X X^T + Y + i\Omega_m - X \Sigma_n \left(\cosh(2r) I_{2n} - i\Omega_n \right) \Sigma_n X^T \qquad (**)$$

> 0

> 0

Consider that 2nd term

$$= -\cosh(2r) X X^T + i X \Sigma_n \Omega_n \Sigma_n X^T$$

$$\Rightarrow = -\cosh(2r)\, X X^T$$

$$+ iX (I_n \otimes \sigma_z)(I_n \otimes \Lambda_1)(I_n \otimes \sigma_z) X^T$$

$$= -\cosh(2r)\, X X^T \qquad (\text{using } \sigma_z \Lambda_1 \sigma_z = -\Lambda_1)$$

$$- iX \Lambda_n X^T$$

$$\Rightarrow (**) = Y + i\Lambda_m - iX\Lambda_n X^T > 0$$

$$\Rightarrow \quad Y + i\Lambda_m > iX\Lambda_n X^T$$

Then invoking Schur comp. generalisation for PSD, we get

$$Y + i\Lambda_m \geq iX\Lambda_n X^T$$

Wigner function of a quantum channel

$$W_{\mathcal{N}}(r'|r) = (2\pi)^n \, \mathrm{Tr}\left[\hat{A}_{r'} \, \mathcal{N}(\hat{A}_r)\right]$$

where $\hat{A}_r = \hat{D}_r \hat{A}_0 D_{-r}$

$$\hat{A}_0 = \frac{1}{(2\pi)^{2n}} \int dr' \, \hat{D}_{-r'}$$

Using definitions,
this then reduces to

$$\frac{(2\pi)^n}{(2\pi)^{4n}} \int dr'' \int dr''' \, e^{i(r'')^T \Omega r} \, e^{i(r''')^T \Omega r'} \, \mathrm{Tr}\left[\hat{D}_{-r'''} \, \mathcal{N}(\hat{D}_{-r''})\right]$$

Using $\mathrm{Tr}\left[\hat{D}_{-r'''} \, \mathcal{N}(\hat{D}_{-r''})\right]$

$$= \mathrm{Tr}\left[\mathcal{N}^\dagger(\hat{D}_{-r'''}) \, \hat{D}_{-r''}\right]$$

& $\mathcal{N}^\dagger(\hat{D}_z) = \hat{D}_{\Omega X^T \Omega^T z} \, e^{-\frac{1}{4} z^T \Omega Y \Omega^T z + i z^T \Omega \delta}$

for Gaussian channel $\mathcal{N}$ characterized by scaling X, noise Y, & shift δ

$\Rightarrow$ $W_{\mathcal{N}}(r'|r) =$

$$\frac{(2\pi)^n}{(2\pi)^{4n}} \iint dr'' dr''' \; e^{i\left[(r'')^T \Omega r + (r''')^T \Omega r'\right]}$$

$$\mathrm{Tr}\left\{ \hat{D}_{-\Omega X^T \Omega^T r'''} \, \hat{D}_{-r''} \right\}$$

$$e^{-\frac{1}{4}(r''')^T \Omega Y \Omega^T r''' - i (r''')^T \Omega \delta}$$

using $\mathrm{Tr}\{\hat{D}_{r_1} \hat{D}_{-r_2}\} = (2\pi)^2 \delta^{(2n)}(r_1 - r_2)$

$\Rightarrow$ $W_{\mathcal{N}}(r'|r) =$

$$\frac{1}{(2\pi)^{2n}} \iint dr'' dr''' \; e^{i\left[(r'')^T \Omega r + (r''')^T \Omega r'\right]}$$

$$\delta^{(2n)}\left(r'' + \Omega X^T \Omega^T r'''\right)$$

$$e^{-\frac{1}{4}(r''')^T \Omega Y \Omega^T r''' - i (r''')^T \Omega \delta}$$

$$= \frac{1}{(2\pi)^{2n}} \int dr''' \; e^{i\left[-(\Omega X^T \Omega^T r''')^T \Omega r - (r')^T \Omega r'''\right]}$$

$$e^{-\frac{1}{4}(r''')^T \Omega Y \Omega^T r''' + i\delta^T \Omega r'''}$$

$$= \frac{1}{(2\pi)^{2n}} \int dr''' \; e^{-i\left[(r''')^T \Omega X r - (r')^T \Omega r'''\right.}$$

$$e^{-\frac{1}{4}(r''')^T \Omega Y \Omega^T r''' + i\delta^T \Omega r'''}$$

$$= \frac{1}{(2\pi)^{2n}} \int dr''' \; e^{i\left([Xr - r' + \delta]^T \Omega r'''\right)}$$

$$e^{-\frac{1}{4}(r''')^T \Omega Y \Omega^T r'''}$$

rewrite again as

$$= \frac{1}{(2\pi)^{2n}} \int dr''' \; e^{(r''')^T\left[\Omega^T i (Xr - r' + \delta)\right]}$$

$$e^{-(r''')^T\left(\frac{1}{4}\Omega Y \Omega^T\right) r'''}$$

$$A = \frac{1}{4}\Omega Y \Omega^T$$

$$b = i\Omega^T(Xr - r' + \delta)$$

Use Gaussian integration formula

$$= \frac{1}{(2\pi)^{2n}} \frac{\pi^n}{\sqrt{\mathrm{Det}(\frac{1}{4}\Omega Y \Omega^T)}} \times$$

$$e^{\frac{1}{4}(i\Omega^T(Xr - r' + \delta))^T (\frac{1}{4}\Omega Y \Omega^T)^{-1} \cdot (i\Omega^T(Xr - r' + \delta))}$$

$$= \frac{1}{\pi^n \sqrt{\mathrm{Det}(Y)}} e^{-(Xr - r' + \delta)^T Y^{-1} (Xr - r' + \delta)}$$

then conclude that

$$W_N(r'|r) =$$

$$\frac{1}{\pi^n \sqrt{\mathrm{Det}(Y)}} e^{-(r' - (Xr + \delta))^T Y^{-1} (r' - (Xr + \delta))}$$

this is the same as the conditional prob. dist. for a classical Gaussian channel