Lecture 17

Last time, we proved that a $2n \times 2n$ symplectic matrix has a singular value decomposition as

$$S = O_1 Z O_2$$

where O_1 & O_2 are symplectic & orthogonal

& $Z = \bigoplus_{j=1}^{n} \begin{bmatrix} z_j & 0 \\ 0 & z_j^{-1} \end{bmatrix}$

where $z_j \geq 1$

This time, we argue that this decomposition has an important physical interpretation as a method for implementing any unitary corresponding to a symplectic transformation in terms of a passive ~~transform~~ linear-optical transformation active squeezing transformations, & a passive linear-optical transformation.

Why do O_1 & O_2 correspond to passive transformations?

They do not change the mean photon number of the state on which they act.

Recall that $\hat{n} = \frac{1}{2}(\hat{x}^2 + \hat{p}^2 - 1)$

For a single-mode state ρ, the mean photon number is given by $\langle \hat{n} \rangle_\rho = \frac{1}{2}\left(\langle \hat{x}^2 \rangle_\rho + \langle \hat{p}^2 \rangle_\rho - 1 \right)$

Now consider the following relation

$$\sigma_{jk} + 2\bar{r}_j \bar{r}_k = \mathrm{Tr}\left[\{\hat{r}_j, \hat{r}_k\}\rho\right]$$

$$\Rightarrow \sigma_{jj} + 2\bar{r}_j^2 = 2\mathrm{Tr}\left[\hat{r}_j^2 \rho\right] = 2\langle \hat{r}_j^2 \rangle_\rho$$

$$\Rightarrow \langle \hat{n} \rangle_\rho = \frac{1}{2}\left(\frac{\sum_{j=1}^{2} \sigma_{jj}}{2} + \sum_{j=1}^{2} \bar{r}_j^{\,2} - 1 \right)$$

$$= \frac{1}{2}\left(\frac{\mathrm{Tr}[\sigma]}{2} + \bar{r}^T \bar{r} - 1 \right)$$

Generalizing this to n modes gives

$$\langle \hat{N} \rangle_\rho = \frac{1}{2}\left(\frac{\mathrm{Tr}[\sigma]}{2} + \bar{r}^T \bar{r} - n \right)$$

where $\hat{N} = \sum_{j=1}^{n} \hat{n}_j$ is the total photon number operator

Since a symplectic orthogonal transformation O has the following effect

$\bar{r} \to O\bar{r}$

$\sigma \to O \sigma O^T$

& $\langle \hat{N} \rangle$ is invariant under this change, it follows that O is passive.

~~Any~~ Some simple passive transformations are the single-mode phase shifter

$$\rho \to e^{-i\hat{n}\phi}\, \rho\, e^{i\hat{n}\phi} \quad (\text{w/ Hamiltonian } \phi\hat{n})$$

$$\equiv \hat{S}^{\dagger}\, \rho\, \hat{S} \qquad \text{for } \phi \in [0, 2\pi]$$

Using again that $\hat{n} = \frac{1}{2}(\hat{x}^2 + \hat{p}^2 - 1)$

$$= \frac{1}{2}(\hat{x}^2 + \hat{p}^2) - \frac{1}{2}$$

$$= \frac{1}{2}\hat{r}^T\hat{r} - \frac{1}{2}$$

$$\Rightarrow e^{-i\hat{n}\phi} = e^{-i\frac{\phi}{2}\hat{r}^T\hat{r}}\, \underbrace{e^{-i\phi/2}}_{\text{irrelevant global phase}} \simeq e^{-i\frac{\phi}{2}\hat{r}^T\hat{r}}$$

$\Rightarrow$ Hamiltonian matrix is $H = \phi I$

$\Rightarrow$ symplectic transform is $R_\phi = e^{\Omega\phi} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix}$

the Hamiltonian matrix for a beam splitter is given by

$$H_{BS} = \theta \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \hat{H} = \frac{1}{2} \hat{r}^T H_{BS} \hat{r} = \hat{p}_1 \hat{x}_2 - \hat{x}_1 \hat{p}_2$$

$$= i\left(\hat{a}_1^\dagger \hat{a}_2 - \hat{a}_1 \hat{a}_2^\dagger\right)$$

$\Rightarrow$ symp. transform realized is

$$\begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & \cos\theta & 0 & \sin\theta \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

More general beamsplitter transformation is given by

$$\hat{H}_{BS}(\theta, \varphi) = \theta\left(e^{i\varphi} \hat{a}_1^\dagger \hat{a}_2 + e^{-i\varphi} \hat{a}_1 \hat{a}_2^\dagger\right)$$

$$= \theta\left[\cos\varphi \left(\hat{x}_1 \hat{x}_2 + \hat{p}_1 \hat{p}_2\right) + \sin\varphi \left(\hat{p}_1 \hat{x}_2 - \hat{x}_1 \hat{p}_2\right)\right]$$

& Hamiltonian matrix is

$$H_{BS}(\theta,\varphi) = \theta \begin{bmatrix} 0 & 0 & \cos\varphi & -\sin\varphi \\ 0 & 0 & \sin\varphi & \cos\varphi \\ \cos\varphi & \sin\varphi & 0 & 0 \\ -\sin\varphi & \cos\varphi & 0 & 0 \end{bmatrix}$$

& symp. transformation is

$$\begin{bmatrix} \begin{matrix} \cos\theta & 0 \\ 0 & \cos\theta \end{matrix} & \begin{bmatrix} \sin\theta \ \mathrm{rot}(\phi) \end{bmatrix} \\ \begin{bmatrix} -\sin\theta \ \mathrm{rot}(\phi)^T \end{bmatrix} & \begin{matrix} \cos\theta & 0 \\ 0 & \cos\theta \end{matrix} \end{bmatrix}$$

where $\mathrm{rot}(\phi) = \begin{bmatrix} \sin\phi & \cos\phi \\ -\cos\phi & \sin\phi \end{bmatrix}$

effect on annihilation operators is

$$\begin{bmatrix} \cos\theta & -ie^{i\varphi}\sin\theta \\ -ie^{-i\varphi}\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \hat{a}_1 \\ \hat{a}_2 \end{bmatrix}$$

Isomorphism between $2n \times 2n$ symp. orthogonal matrices & $n \times n$ unitary matrices (7)

Suppose that a symplectic matrix S is orthogonal

Writing $S = \begin{bmatrix} X & Y \\ W & Z \end{bmatrix}$

and taking symplectic form as $J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$

then symplectic condition $SJS^T = J$ imposes that

$$-YX^T + XY^T = -ZW^T + WZ^T = 0$$

$$-YW^T + XZ^T = I$$

& $S^T J S = J$

$$\Rightarrow \quad -W^T X + X^T W = -Z^T Y + Y^T Z = 0$$

$$X^T Z - W^T Y = I$$

orthogonality ~~implies that~~

$$S^T S = S S^T = I \Rightarrow \quad XX^T + YY^T = WW^T + ZZ^T = I$$

$$XW^T + YZ^T = 0$$

$$\& \; X^TX + W^TW = Y^TY + Z^TZ = I$$

$$X^TY + W^TZ = 0$$

can conclude from these conditions that

$$Z = X \; \& \; W = -Y$$

$\Rightarrow$ any symplectic orthogonal matrix has the form

$$S = \begin{pmatrix} X & Y \\ -Y & X \end{pmatrix} \quad \text{w/ } XY^T - YX^T = 0 \;\; \& \;\; XX^T + YY^T = I$$

At the same time, any $n \times n$ unitary matrix can be written as $X + iY$ for real $X \& Y$

$\& \; UU^+$

$$= \underbrace{XX^T + YY^T}_{I} + i\underbrace{(YX^T - XY^T)}_{0} = I$$

so this gives isomorphism between $n \times n$ unitaries $\&$ $2n \times 2n$ symp. orth. matrices

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We can also block diagonalize S

by $\bar{u}' = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \otimes I_n \equiv u' \otimes I_n$

since $S = I_2 \otimes X + \Lambda_1 \otimes Y$

& $u' \Lambda_1 u'^{\dagger} = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}$

$\Rightarrow \bar{u}' S \bar{u}'^{\dagger}$

$= (u' \otimes I_n)(I_2 \otimes X + \Lambda_1 \otimes Y)(u' \otimes I_n)^{\dagger}$

$= (u' \otimes I_n)(I_2 \otimes X)(u' \otimes I_n)^{\dagger} +$

$(u' \otimes I_n)(\Lambda_1 \otimes Y)(u' \otimes I_n)^{\dagger}$

$= I_2 \otimes X + \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} \otimes Y$

complex conj. of entries

$= \begin{bmatrix} X - iY & 0 \\ 0 & X + iY \end{bmatrix} = \begin{bmatrix} \bar{U}^{*} & 0 \\ 0 & U \end{bmatrix}$

So an orthogonal transformation of quadrature operators is the same as a unitary transformation of vector of annihilation operators:

$$U \begin{bmatrix} \hat{a}_1 \\ \vdots \\ \hat{a}_n \end{bmatrix} = U \underline{\hat{a}}$$

Since total photon number operator is

$$\hat{N} = \sum_{j=1}^{n} \hat{a}_j^\dagger \hat{a}_j = \underline{\hat{a}}^\dagger \underline{\hat{a}}$$

then unitary action on $\underline{\hat{a}}$ does not change $\hat{N}$ b/c

$$(U\underline{\hat{a}})^\dagger U\underline{\hat{a}} = \underline{\hat{a}}^\dagger U^\dagger U \underline{\hat{a}} = \underline{\hat{a}}^\dagger \underline{\hat{a}}$$

discuss algorithm of Clements et al. from 1603.08788 using projector

Squeezing transformations ~~ap~~ have symplectic matrices of the form

$$S_{SQ} = \begin{pmatrix} z_j & 0 \\ 0 & z_j^{-1} \end{pmatrix}$$

can get Hamiltonian matrix via

$$H = \Omega^T \ln S_{SQ}$$

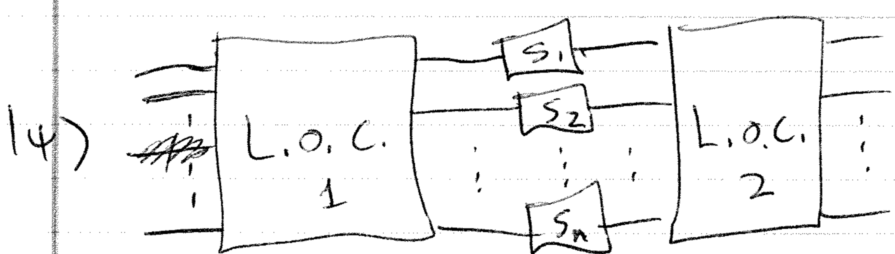
$$= \ln z \cdot \sigma_x \equiv \begin{bmatrix} 0 & \ln z \\ \ln z & 0 \end{bmatrix}$$

then Hamiltonian operator $\hat{H}$ is

$$\hat{H} = \frac{1}{2} \hat{r}^T H \hat{r} = \frac{\ln z}{2} \{\hat{x}_j, \hat{p}_j\} = \frac{\ln z}{2} (\hat{x}_j \hat{p}_j + \hat{p}_j \hat{x}_j)$$

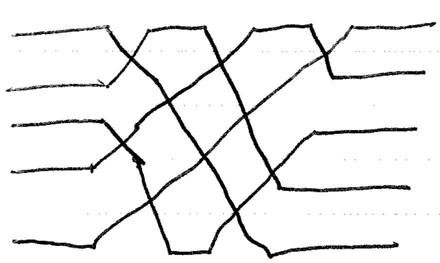
$$= -\frac{i \ln z}{2} \left(\hat{a}_j^2 - (\hat{a}_j^\dagger)^2\right)$$

In summary, a general Gaussian unitary corresponding to a symplectic matrix S can be implemented as



L.O.C – linear-optic circuit, passive transformation

& each L.O.C has implementation



e.g., for 5 modes

is a beam splitter