Lecture 15

Characteristic function of a Gaussian state

Recall that a generic Gaussian state ρ can be written as

$$\rho = \hat{D}_{-\bar{r}} \hat{S} \left[\bigotimes_{j=1}^{n} \theta_{A_j}(\bar{n}_j) \right] \hat{S}^{\dagger} \hat{D}_{\bar{r}}$$

then characteristic function is

$$\chi_{\rho}(\underline{r}) = \mathrm{Tr}[\hat{D}_{-\underline{r}} \, \rho]$$

$$= \mathrm{Tr}\left[\hat{D}_{-\underline{r}} \hat{D}_{-\bar{r}} \hat{S} \bigotimes_{j=1}^{n} \theta_{A_j}(\bar{n}_j) \hat{S}^{\dagger} \hat{D}_{\bar{r}}\right]$$

$$= \mathrm{Tr}\left[\hat{S}^{\dagger} \hat{D}_{\bar{r}} \hat{D}_{-\underline{r}} \hat{D}_{-\bar{r}} \hat{S} \bigotimes_{j=1}^{n} \theta_{A_j}(\bar{n}_j)\right]$$

$$= e^{i \bar{r}^{T} \Omega \underline{r}} \, \mathrm{Tr}\left\{\hat{S}^{\dagger} \hat{D}_{\bar{r}} \hat{D}_{-\bar{r}} \hat{D}_{-\underline{r}} S \bigotimes_{j=1}^{n} \theta_{A_j}(\bar{n}_j)\right\}$$

$$= e^{i\hat{r}^T \Lambda \underline{r}} \quad \mathrm{Tr}\left[\hat{S}^\dagger \hat{D}_{-\underline{r}} \hat{S} \hat{\bigotimes_{j=1}} \theta_{A_j}(\bar{n}_j) \right]$$

Now using that

$$\hat{S}^\dagger \hat{D}_{-\underline{r}} \hat{S} =$$

$$e^{-\frac{i}{2}\hat{r}^T H \hat{r}} \; e^{-i\underline{r}^T \Lambda \hat{r}} \; e^{+\frac{i}{2}\hat{r}^T H \hat{r}}$$

$$= \exp\left(-i\underline{r}^T \Lambda \, \hat{S}^\dagger \hat{r} \hat{S}\right)$$

$$= \exp\left(-i\underline{r}^T \Lambda \, e^{-\Lambda H} \hat{r}\right)$$

$$= \exp\left(-i\underline{r}^T \Lambda \, \overset{\equiv \text{ defn}}{S^{-1}} \hat{r}\right)$$

$$= \exp\left(-i\underline{r}^T S^{\dagger T} \Lambda \hat{r}\right)$$

$$= \exp\left(-i (S\underline{r})^T \Lambda \hat{r}\right)$$

$$= \hat{D}_{-S\underline{r}}$$

$$\Rightarrow \chi_\rho(\underline{r}) = e^{i\bar{r}^T \Omega \underline{r}} \; \mathrm{Tr}\left[\hat{D}_{-S\underline{r}} \bigotimes_{j=1}^{n} \theta_{A_j}(\bar{n}_j)\right]$$

Now consider that

$$\theta(\bar{n}) = \frac{1}{\bar{n}+1} \sum_{n=0}^{\infty} \left(\frac{\bar{n}}{\bar{n}+1}\right)^n |n\rangle\langle n|$$

(exercise) $$= \frac{1}{\pi \bar{n}} \int d^2\alpha \; e^{-|\alpha|^2/\bar{n}} \; |\alpha\rangle\langle\alpha|$$

$$\Rightarrow \mathrm{Tr}\left[\hat{D}_\alpha \, \theta(\bar{n})\right] = \chi_{\theta(\bar{n})}(\alpha)$$

$$= \frac{1}{\pi \bar{n}} \int d^2\beta \; e^{-|\beta|^2/\bar{n}} \; \mathrm{Tr}\left[\hat{D}_\alpha \, |\beta\rangle\langle\beta|\right]$$

$$= \frac{1}{\pi \bar{n}} \int d^2\beta \; e^{-|\beta|^2/\bar{n}} \; \langle\beta|\hat{D}_\alpha|\beta\rangle$$

$$= \frac{1}{\pi \bar{n}} \int d^2\beta \; e^{-|\beta|^2/\bar{n}} \; e^{-\frac{|\alpha|^2}{2}} \cdot e^{\frac{1}{2}\left((\alpha+\beta)\beta^* - (\alpha+\beta)^*\beta\right)} \cdot e^{\frac{1}{2}(\alpha\beta^* - \alpha^*\beta)}$$

$$= \frac{1}{\pi \bar{n}} \int d^2\beta \; e^{-|\beta|^2/\bar{n}} \; e^{-|\alpha|^2/2} \cdot e^{\alpha\beta^* - \alpha^*\beta}$$

$$= \underbrace{\frac{e^{-|\alpha|^2/2}}{\pi \bar{n}} \int d^2\beta \; e^{-|\beta|^2/\bar{n}} \, e^{\alpha\beta^* - \alpha^*\beta}}$$

complex Fourier transform of $\frac{1}{\pi\bar{n}} * e^{-|\beta|^2/\bar{n}}$

$$= e^{-|\alpha|^2/2} \cdot e^{-|\alpha|^2 \bar{n}}$$

$$= e^{-\frac{|\alpha|^2}{2}(2\bar{n}+1)}$$

$$= e^{-\frac{|\alpha|^2}{2}\nu}$$

← "nu"

↑ symplectic eigenvalue.

Now substituting $\alpha = \frac{x+ip}{\sqrt{2}}$,

we get

$$\chi_{\Theta(\bar{n})}(\underline{r}) = e^{-\frac{1}{4}(x^2+p^2)\nu}$$

$$= e^{-\frac{1}{4}\underline{r}^T \nu \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \underline{r}}$$

↑ "nu"

$$\Rightarrow \chi_{\bigotimes_{j=1}^{n}\theta(\bar{n}_j)}(\underline{r}) = e^{-\frac{1}{4}\underline{r}^T \bigoplus_{j=1}^{n} r_j \begin{bmatrix}1&0\\0&1\end{bmatrix} \underline{r}}$$

$$\Rightarrow \mathrm{Tr}\left[\hat{D}_{-S\underline{r}} \bigotimes_{j=1}^{n} \theta_{A_j}(\bar{n}_j)\right]$$

$$= \chi_{\bigotimes_{j=1}^{n}\theta(\bar{n}_j)}(S\underline{r})$$

$$= e^{-\frac{1}{4}(S\underline{r})^T \bigoplus_{j=1}^{n} r_j \begin{bmatrix}1&0\\0&1\end{bmatrix} S\underline{r}}$$

Let $D = \bigoplus_{j=1}^{n} r_j \begin{bmatrix}1&0\\0&1\end{bmatrix}$

$= \underline{D} \otimes I_2$

where $\underline{D} = \begin{bmatrix} r_1 & & & \\ & r_2 & & \\ & & \ddots & \\ & & & r_n \end{bmatrix}$

Then $\Lambda = I_n \otimes \Lambda_1$

$$\Rightarrow = e^{-\frac{1}{4}\underline{r}^T S^{+T}(\underline{D}\otimes I_2) S \underline{r}}$$

$$= e^{-\frac{1}{4}\underline{r}^T S^{+T}(\underline{D}\otimes I_2)(I_n\otimes\Lambda_1)^T(I_n\otimes\Lambda_1) S \underline{r}}$$

$$= e^{-\frac{1}{4}\underline{r}^T S^{+T} (I_n \otimes \Omega_1)^T (\underline{D} \otimes I_2)(I_n \otimes \Omega_1) S \underline{r}}$$

$$= e^{-\frac{1}{4}\underline{r}^T S^{+T} \Omega^T D \Omega S \underline{r}}$$

$$= e^{-\frac{1}{4}\underline{r}^T \Omega^T \underbrace{S^{-1} D S^{-T}}_{\sigma} \Omega \underline{r}}$$

$$= e^{-\frac{1}{4}\underline{r}^T \Omega^T \sigma \Omega \underline{r}}$$

$$\Rightarrow \chi_\rho(\underline{r}) = e^{-\frac{1}{4}\underline{r}^T \Omega^T \sigma \Omega \underline{r}} \; e^{i \bar{r}^T \Omega \underline{r}}$$

cov. matrix   mean vector

so then characteristic function of Gaussian state depends only on mean vector $\bar{r}$ + cov. matrix σ.

using representation

$$\rho = \frac{1}{(2\pi)^n} \int d\underline{r}\; \chi_\rho(\underline{r})\, \hat{D}_{\underline{r}}$$

This means that we can write a Gaussian state as

$$\rho = \frac{1}{(2\pi)^n}\int d\underline{r}\ e^{-\frac{1}{4}\underline{r}^T\Omega^T\sigma\Omega\underline{r}}\ e^{i\bar{r}^T\Omega\underline{r}}\ \hat{D}_{\underline{r}}$$

Defining $\tilde{r} = \Omega\underline{r}$ & noting that $\det(\Omega)=1$

$$\Rightarrow \quad \rho = \frac{1}{(2\pi)^n}\int d\tilde{r}\ e^{-\frac{1}{4}\tilde{r}^T\sigma\tilde{r} + i\bar{r}^T\tilde{r}}\ \hat{D}_{\Omega^T\tilde{r}}$$

Recalling that Wigner function

$$W_\rho(\underline{r}) = \frac{1}{(2\pi)^{2n}}\int d\underline{r}'\ e^{i\underline{r}'^T\Omega\underline{r}}\ \chi_\rho(\underline{r}')$$

$\Rightarrow$ for a Gaussian state

$$W_\rho(\underline{r}) = \frac{1}{(2\pi)^{2n}}\int d\underline{r}'\ e^{-\frac{1}{4}\underline{r}'^T\Omega^T\sigma\Omega\underline{r}'}\ e^{i\bar{r}^T\Omega\underline{r}'}\ e^{i\underline{r}'^T\Omega\underline{r}}$$

$$= \frac{1}{(2\pi)^{2n}} \int d\underline{r}' \, e^{-\frac{1}{4} \underline{r}'^T \Lambda^T \sigma \Lambda \underline{r}'} \, e^{i \underline{r}'^T \Lambda^T (\bar{r} - \underline{r})}$$

Now use the fact that

$$\int d\underline{r} \; e^{-\underline{r}^T A \underline{r} + \underline{r}^T b} =$$

$$\frac{\pi^n}{\sqrt{\mathrm{Det}(A)}} \; e^{\frac{1}{4} b^T A^{-1} b}$$

w/ $A = \frac{1}{4} \Lambda^T \sigma \Lambda$

$b = i \Lambda^T (\bar{r} - \underline{r})$

$$\Rightarrow = \frac{1}{(2\pi)^{2n}} \frac{\pi^n}{\sqrt{\mathrm{Det}(\frac{1}{4} \Lambda^T \sigma \Lambda)}} \cdot \cancel{e^{\frac{1}{4}(i\Lambda^T(\bar{r}-\underline{r}))^T}}$$

$$e^{\frac{1}{4} (i \Lambda^T (\bar{r} - \underline{r}))^T (\frac{1}{4} \Lambda^T \sigma \Lambda)^{-1} (i \Lambda^T (\bar{r} - \underline{r}))}$$

$$= \frac{1}{\pi^n \sqrt{\mathrm{Det}(\sigma)}} \, e^{-(\underline{r} - \bar{r})^T \sigma^{-1} (\underline{r} - \bar{r})}$$

Defining $\sigma' = \frac{\sigma}{2}$

this becomes

$$\underbrace{\frac{1}{\sqrt{(2\pi)^{2n}\,\mathrm{Det}(\sigma')}}\, e^{-\frac{1}{2}(\underline{r}-\bar{r})^T(\sigma')^{-1}(\underline{r}-\bar{r})}}$$

which is the same as a
2n-dimensional multivariate
normal prob. dens. function.

So every Gaussian state
has a Gaussian Wigner
function, which is a
prob. dens. function.

Note: development holds for pure or
non-faithful Gaussian states since we end up
w/ σ at end.

Can now revisit overlap formulla for Gaussian states ρ_1 & ρ_2:

Consider that

$$\mathrm{Tr}\{\rho_1 \rho_2\}$$

$$= \mathrm{Tr}\left[\frac{1}{(2\pi)^n}\int dr_1\, e^{-\frac{1}{4} r_1^T \Lambda^T \sigma_1 \Lambda r_1 + i r_1^T \Lambda^T \bar{r}_1}\, \hat{D}_{r_1} \cdot \right.$$

$$\left. \frac{1}{(2\pi)^n}\int dr_2\, e^{-\frac{1}{4} r_2^T \Lambda^T \sigma_2 \Lambda r_2 - i r_2^T \Lambda^T \bar{r}_2}\, \hat{D}_{-r_2}\right]$$

$$= \frac{1}{(2\pi)^{2n}} \iint dr_1\, dr_2\, e^{-\frac{1}{4} r_1^T \Lambda^T \sigma_1 \Lambda r_1 + i r_1^T \Lambda^T \bar{r}_1} \cdot e^{-\frac{1}{4} r_2^T \Lambda \sigma_2 \Lambda r_2 - i r_2^T \Lambda^T \bar{r}_2} \cdot \underbrace{\mathrm{Tr}[\hat{D}_{r_1} \hat{D}_{-r_2}]}_{(2\pi)^n \delta^{(2n)}(r_1 - r_2)}$$

$$= \frac{1}{(2\pi)^n}\int dr\, e^{-\frac{1}{4} r^T \Lambda^T (\sigma_1 + \sigma_2) \Lambda r + i r^T \Lambda^T (\bar{r}_1 - \bar{r}_2)}$$

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reuse previous calculation to get

$$= \frac{2^n}{\sqrt{\mathrm{Det}(\sigma_1+\sigma_2)}} e^{-(\bar{r}_1-\bar{r}_2)^T(\sigma_1+\sigma_2)^{-1}(\bar{r}_1-\bar{r}_2)}$$

same formula calculated previously, but now for Gaussian states w/ non-zero mean vectors.

Dynamics

Gaussian unitary transformations

Similar to how faithful Gaussian states can be written as

$$\propto e^{-\frac{1}{2}(\hat{r}-\bar{r})^T H (\hat{r}-\bar{r})}$$

where Hamiltonian is

$$\hat{H} = \frac{1}{2}(\hat{r}-\bar{r})^T H (\hat{r}-\bar{r})$$

we can write quadratic unitary evolutions as

$$e^{-i\hat{H}}$$

$$= e^{-\frac{i}{2}(\hat{r}-\bar{r})^T H (\hat{r}-\bar{r})}$$

From the fact that

$$\hat{H} = \hat{D}_{-\bar{r}} \left(\frac{1}{2}\hat{r}^T H \hat{r}\right) \hat{D}_{\bar{r}}$$

We can write

$$e^{-i\hat{H}} = \hat{D}_{-\bar{r}}\; e^{-\frac{i}{2}\hat{r}^T H \hat{r}}\; \hat{D}_{\bar{r}}$$

Define $\hat{S} = e^{\frac{i}{2}\hat{r}^T H \hat{r}}$

Then $\hat{S}\hat{r}\hat{S}^{+} = S\hat{r}$

where $S = e^{\Omega H}$

Then

$$\hat{S}\,\hat{D}_{\bar{r}}\,S^{+}$$

$$= \hat{S}\, e^{i\bar{r}^T\Omega\hat{r}}\, S^{+}$$

$$= e^{i\bar{r}^T\Omega(\hat{S}\hat{r}\hat{S}^{+})} = e^{i\bar{r}^T\Omega S\hat{r}}$$

$$= e^{i\bar{r}^T S^{-T}\Omega\hat{r}} = e^{i(S^{-1}\bar{r})^T\Omega\hat{r}}$$

$$= \hat{D}_{S^{-1}\bar{r}}$$

$$\Rightarrow \quad \hat{S}\hat{D}_{\bar{r}}\hat{S}^{+} = \hat{D}_{S^{-1}\bar{r}}$$

$$\Rightarrow \quad \hat{S}\hat{D}_{\bar{r}} = \hat{D}_{S^{-1}\bar{r}}\hat{S}$$

(commutation rule for displacement op. + purely quadratic Ham.)

$$\Rightarrow \quad \hat{S}^{+}\hat{D}_{\bar{r}} = \hat{D}_{S\bar{r}}\hat{S}^{+}$$

$$\Rightarrow \quad e^{-i\hat{H}} = \hat{D}_{-\bar{r}}\hat{S}^{+}\hat{D}_{\bar{r}}$$

$$= \hat{D}_{-\bar{r}}\hat{D}_{S\bar{r}}\hat{S}^{+}$$

$$= e^{i\bar{r}^{T}\Omega S\bar{r}/2}\hat{D}_{(S-I)\bar{r}}\hat{S}^{+}$$

$\Rightarrow$ Gaussian unitary can be written as purely quadratic evolution followed by purely linear evolution.

All of the above implies that a quadratic evolution evolves a Gaussian state input to another Gaussian state.

- Consider that

$$\rho_1 \propto \hat{D}_{-\bar{r}_1}\, \hat{S}_1 \left[\bigotimes_{j=1}^{n} \theta(\bar{n}_j) \right] \hat{S}_1^{\dagger}\, \hat{D}_{\bar{r}_1}$$

$$e^{-i\hat{H}} \rho_1 e^{+i\hat{H}}$$

$$= \underbrace{\hat{D}_{(S-I)\bar{r}}\, \hat{S}^{\dagger}\, \hat{D}_{-\bar{r}_1}\, \hat{S}_1}\left[\bigotimes_{j=1}^{n} \theta(\bar{n}_j) \right] H.C.$$

can then use commutation rules to get $\hat{D}_{-\bar{r}'}\, \hat{S}' \left[\bigotimes_{j=1}^{n} \theta(\bar{n}_j) \right. \hat{S}'^{\dagger}\, \hat{D}_{\bar{r}'}$

Now suppose that a unitary U takes one Gaussian state ρ_1 to another Gaussian state ρ_2

$$\rho_2 = U \rho_1 U^+$$

the transformation U does not change eigenvalues of ρ_1

So if $\rho_1 = \hat{D}_{-\bar{r}_1} \hat{S}_1 \left[\bigotimes_{j=1}^{n} \theta(\bar{n}_j) \right] \hat{S}_1^+ \hat{D}_{\bar{r}_1}$

then $\rho_2 = \hat{D}_{-\bar{r}_2} \hat{S}_2 \left[\bigotimes_{j=1}^{n} \theta(\bar{n}_j) \right] \hat{S}_2^+ \hat{D}_{\bar{r}_2}$

But then from this, we infer that

$$\underbrace{U = \hat{D}_{-\bar{r}_2} \hat{S}_2 \hat{S}_1^+ \hat{D}_{\bar{r}_1}}$$

this is then a Gaussian (quadratic) evolution