Lecture 14

(1)

What was left to prove from last time is that

$$|0\rangle\langle 0| = \frac{1}{\pi}\int_{\mathbb{C}} d^2\gamma \; e^{-\frac{1}{2}|\gamma|^2}\hat{D}_\gamma$$

to prove equality, again look at matrix elements in overcomplete coherent-state basis.

Then $\langle\delta|0\rangle\langle 0|\varepsilon\rangle = e^{-\frac{1}{2}(|\delta|^2+|\varepsilon|^2)}$

For the RHS,

$\langle\delta|$ " $|\varepsilon\rangle$

$$= \frac{1}{\pi}\int_{\mathbb{C}} d^2\gamma \; e^{-\frac{1}{2}|\gamma|^2}\langle\delta|\hat{D}_\gamma|\varepsilon\rangle$$

$$= \frac{1}{\pi}\int_{\mathbb{C}} d^2\gamma \; e^{-\frac{1}{2}|\gamma|^2}\langle\delta|\hat{D}_\gamma\hat{D}_\varepsilon|0\rangle$$

$$= \frac{1}{\pi}\int_{\mathbb{C}} d^2\gamma \; e^{-\frac{1}{2}|\gamma|^2}\langle\delta|\hat{D}_{\gamma+\varepsilon}|0\rangle \, e^{\frac{1}{2}(\gamma\varepsilon^*-\gamma^*\varepsilon)}$$

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$$= \frac{1}{\pi} \int_{\mathbb{C}} d^2\gamma \; e^{-\frac{1}{2}|\gamma|^2} \langle \delta | \gamma + \varepsilon \rangle \, e^{\frac{1}{2}(\gamma \varepsilon^* - \gamma^* \varepsilon)}$$

$$= \frac{1}{\pi} \int_{\mathbb{C}} d^2\gamma \; e^{-\frac{1}{2}|\gamma|^2} \, e^{-\frac{1}{2}|\gamma + \varepsilon - \delta|^2} \, e^{\frac{1}{2}\left((\gamma+\varepsilon)\delta^* - (\gamma+\varepsilon)^*\delta \right)} \, e^{\frac{1}{2}(\gamma \varepsilon^* - \gamma^* \varepsilon)}$$

$\vdots$

$$= \frac{1}{\pi} \int_{\mathbb{C}} d^2\gamma \; e^{-\frac{1}{2}|\varepsilon|^2} e^{-\frac{1}{2}|\delta|^2} \cdot e^{-(\gamma+\varepsilon)(\gamma^* - \delta^*)}$$

$$= e^{-\frac{1}{2}\left[|\varepsilon|^2 + |\delta|^2\right]} \underbrace{\frac{1}{\pi} \int d^2\gamma \; e^{-(\gamma+\varepsilon)(\gamma^* - \delta^*)}}_{= \frac{1}{\pi}\int d^2\gamma \, e^{-|\gamma|^2}}$$

to solve this integral, treat γ & γ^* as independent variables. Translation doesn't change integral. So then

$$\Rightarrow \quad = e^{-\frac{1}{2}\left[|\varepsilon|^2 + |\delta|^2\right]}$$

$$\Rightarrow \chi_\rho(\alpha) = \mathrm{Tr}[\hat{D}_\alpha \rho]$$

& $$\rho = \frac{1}{\pi} \int_{\mathbb{C}} d^2\alpha \; \chi_\rho(\alpha) \hat{D}_{-\alpha}$$

$\chi_\rho(\alpha)$ is known as symmetrically ordered characteristic function.

In terms of real variables & for n modes, we have

$$\rho = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{2n}} d\underline{r} \; \chi_\rho(\underline{r}) \hat{D}_{\underline{r}}$$

w/ $d\underline{r} = dx_1 dp_1 \cdots dx_n dp_n$

& $\chi_\rho(\underline{r}) = \mathrm{Tr}[\hat{D}_{-\underline{r}} \rho]$

Properties of characteristic function:

1) $\chi_\rho(0) = 1$ b/c

$$\chi_\rho(0) = \mathrm{Tr}[\hat{D}_0 \rho] = \mathrm{Tr}[\rho] = 1$$

2) Hermiticity $\rho = \rho^\dagger$

$$\Rightarrow \quad \chi_\rho(\underline{r}) = \mathrm{Tr}[\hat{D}_{-\underline{r}}\, \rho]$$

$$= \overline{\mathrm{Tr}\{\hat{D}_{-r}\, \rho\}^*}$$

$$= \overline{\mathrm{Tr}[(\hat{D}_{-r}\, \rho)^\dagger]}$$

$$= \overline{\mathrm{Tr}\{\hat{D}_{\underline{r}}\, \rho\}}$$

$$= \overline{\chi_\rho(-\underline{r})}$$

3) $\rho \geq 0$ (PSD ρ)

Consider that $\rho \geq 0$ iff

$$\mathrm{Tr}\{\rho\, \hat{O}^\dagger \hat{O}\} \geq 0 \quad \forall \text{ operators } \hat{O}$$

Take $\hat{O} = \hat{R}_c = \sum_{j=1}^{\ell} c_j \hat{D}_{r_j}$

where $c_j \in \mathbb{C}$

Then $\mathrm{Tr}[\rho\, \hat{R}_c^\dagger \hat{R}_c]$

$$= \sum_{j,k=1}^{\ell} c_j^* c_k \, \mathrm{Tr}[\rho\, \hat{D}_{r_j}^\dagger \hat{D}_{r_k}] \geq 0$$

Define $D_{jk} = \mathrm{Tr}[\rho\, \hat{D}_{r_j}^\dagger \hat{D}_{r_k}]$

Then the above is

$$c^\dagger D c \geq 0$$

$\forall$ so D should be PSD.

then

$$\mathrm{Tr}\left[\rho\, \hat{D}^{\dagger}_{r_j} \hat{D}_{r_k}\right]$$

$$= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{2n}} dr\ \chi_\rho(r)\ \mathrm{Tr}\left[\hat{D}_r \hat{D}_{-r_j} \hat{D}_{r_k}\right]$$

$$= \frac{1}{(2\pi)^n} e^{i r_k^T \Lambda r_j / 2} \int_{\mathbb{R}^{2n}} dr\, \chi_\rho(r) \cdot \mathrm{Tr}\left[\hat{D}_r \hat{D}_{-(r_j - r_k)}\right]$$

$$= e^{i r_k^T \Lambda r_j / 2}\ \chi_\rho(r_j - r_k)$$

Then we have shown that

$\rho \geq 0 \Rightarrow$

$$D_{jk} = \chi_\rho(r_j - r_k)\, e^{i r_k^T \Lambda r_j / 2}$$

is PSD $\forall$ vectors $r_j \in \mathbb{R}^{2n}$

turns out $\Leftarrow$ holds also

Setting $\Lambda = 0$ gives condition for $\chi(r)$ to be the characteristic function for a classical probability distribution.

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quasi-probability distributions

Take the complex Fourier transform to arrive at quasi-probability dist.:

$$W_\rho(\alpha) = \frac{1}{\pi^2} \int_{\mathbb{C}} d^2\beta \; e^{(\alpha\beta^* - \alpha^*\beta)} \chi_\rho(\beta)$$

or alternatively

$$W_\rho(\underline{r}) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} d\underline{r}' \; e^{i\underline{r}'^T \Omega \underline{r}} \chi_\rho(\underline{r}')$$

Normalized:

$$\int_{\mathbb{C}} d^2\alpha \; W_\rho(\alpha) = \chi_\rho(0) = 1$$

or $$\int_{\mathbb{R}^2} d\underline{r} \; W_\rho(\underline{r}) = 1$$

& has the property that

$$\int_{\mathbb{C}} d^2\alpha \; \alpha^{*m} \alpha^n \; W_\rho(\alpha)$$

$$= \langle \hat{a}^{\dagger m} \hat{a}^n \rangle_W$$

where W means Weyl symmetric ordering.

Wigner function is historically written as, ~~setting~~

$$W_\rho(\underline{r}) = W_\rho(x,p)$$

$$= \frac{1}{2\pi^2} \iint \frac{dx'dp'}{2} \, e^{i(px'-xp')} \chi_\rho(x',p')$$

$$= \frac{1}{(2\pi)^2} \iint dx'dp' \, e^{i(px'-xp')} \int_{\mathbb{R}} dq \langle q | \hat{D}_{-\frac{r'}{2}} \rho \, \hat{D}_{-\frac{r'}{2}} | q \rangle$$

$$= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^3} dq\, dx'\, dp' \; e^{ipx'} e^{ip'(q-x)} \left\langle q + \frac{x'}{2} \right| \rho \left| q - \frac{x'}{2} \right\rangle$$

$$= \frac{1}{2\pi} \int dx' \; e^{ipx'} \left\langle x + \frac{x'}{2} \right| \rho \left| x - \frac{x'}{2} \right\rangle$$

$$= \underbrace{\frac{1}{\pi} \int_{\mathbb{R}} dx' \; e^{i2px'} \langle x + x' | \rho | x - x' \rangle}_{\text{traditional way of writing Wigner function}}$$

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multimode Wigner function is given by

$$W_\rho(\underline{r}) = \frac{1}{(2\pi)^{2n}} \int d\underline{r}' \; e^{i \underline{r}'^T \Omega \underline{r}} \chi_\rho(\underline{r}')$$

can alternatively define it in terms of phase-space point operators

$$\hat{A}_{\underline{r}} = D_{\underline{r}} \hat{A}_0 \hat{D}_{-\underline{r}}$$

where

$$\hat{A}_0 = \frac{1}{(2\pi)^{2n}} \int d\underline{r}' \; \hat{D}_{-\underline{r}'}$$

$$\left(\hat{A}_0 = \frac{1}{(2\pi)^{2n}} \int d\underline{r}' \; \hat{D}_{-\underline{r}'} \right)$$

Then Wigner function is given by

$$W_\rho(\underline{r}) = \mathrm{Tr}[\hat{A}_{\underline{r}} \rho]$$

Why does this hold?

Consider that

$$\hat{D}_{\underline{r}} \hat{D}_{-\underline{r}'} \hat{D}_{-\underline{r}} = e^{i \underline{r}'^T \Omega \underline{r}} \hat{D}_{\underline{r}} \hat{D}_{-\underline{r}} \hat{D}_{\underline{r}'}$$

$$= e^{i \underline{r}'^T \Omega \underline{r}} \hat{D}_{-\underline{r}'}$$

$$\Rightarrow \hat{A}_{\underline{r}} = \frac{1}{(2\pi)^{2n}} \int d\underline{r}' \, \hat{D}_{-\underline{r}'} \, e^{i\underline{r}'^T \Omega \underline{r}}$$

$$\Rightarrow \mathrm{Tr}[\hat{A}_{\underline{r}} \rho]$$

$$= \frac{1}{(2\pi)^{2n}} \int d\underline{r}' \, e^{i\underline{r}'^T \Lambda \underline{r}} \, \mathrm{Tr}[\rho \hat{D}_{-\underline{r}'}]$$

$$= \frac{1}{(2\pi)^{2n}} \int d\underline{r}' \, e^{i\underline{r}'^T \Omega \underline{r}} \, \chi_\rho(\underline{r}')$$

$$= W_\rho(\underline{r})$$

<u>Properties of phase-space point operators</u>

i) Hermitian

$$\hat{A}_{\underline{r}}^\dagger = (D_{\underline{r}} \hat{A}_0 \hat{D}_{-\underline{r}})^\dagger$$

$$= D_{\underline{r}} \hat{A}_0^\dagger \hat{D}_{-\underline{r}}$$

then $$\hat{A}_0^{\dagger} = \frac{1}{(2\pi)^{2n}} \int d\underline{r}' \, (\hat{D}_{-\underline{r}'})^{\dagger}$$

$$= \frac{1}{(2\pi)^{2n}} \int d\underline{r}' \, \hat{D}_{\underline{r}}$$

$$= \hat{A}_0$$ (due to symmetry of integral)

2) Trace $= \frac{1}{(2\pi)^n}$

$$Tr\{\hat{A}_{\underline{r}}\} = Tr[\hat{A}_0]$$

$$= \frac{1}{(2\pi)^{2n}} \int d\underline{r}' \, Tr[\hat{D}_{\underline{r}'}]$$

$$= \frac{1}{(2\pi)^{2n}} \int d\underline{r}' \, (2\pi)^n \delta^{(2n)}(\underline{r}')$$

$$= \frac{1}{(2\pi)^n}$$

3) $\int d\underline{r} \, \hat{A}_{\underline{r}} = \hat{I}$

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$$\int d\underline{r}\, \hat{A}_{\underline{r}} = \frac{1}{(2\pi)^{2n}} \int d\underline{r}'\, \hat{D}_{-\underline{r}'} \int d\underline{r}\, e^{i\underline{r}'^T \Omega \underline{r}}$$

$$= \int d\underline{r}'\, \hat{D}_{-\underline{r}'}\, \delta^{(2n)}(\underline{r}')$$

$$= \hat{D}_0$$

$$= \hat{I}$$

4) Orthogonality

$$\mathrm{Tr}\left[\hat{A}_{\underline{r}_1} \hat{A}_{\underline{r}_2}\right]$$

$$= \mathrm{Tr}\left[\hat{D}_{\underline{r}_1} \hat{A}_0 \hat{D}_{-\underline{r}_1} \hat{D}_{\underline{r}_2} \hat{A}_0 \hat{D}_{-\underline{r}_2}\right]$$

$$= \mathrm{Tr}\left[\hat{A}_0 \hat{D}_{\underline{r}_2-\underline{r}_1} \hat{A}_0 \hat{D}_{-(\underline{r}_2-\underline{r}_1)}\right]$$

$$= \frac{1}{(2\pi)^{4n}} \iint d\underline{r}\, d\underline{r}'\, \mathrm{Tr}\left[\hat{D}_{-\underline{r}'} \hat{D}_{\underline{r}_2-\underline{r}_1} \hat{D}_{\underline{r}}^{*} \hat{D}_{-(\underline{r}_2-\underline{r}_1)}\right]$$

$$= \frac{1}{(2\pi)^{4n}} \iint d\underline{r}\, d\underline{r}' \; e^{i(\underline{r}_2 - \underline{r}_1)^T \Omega \underline{r}} \; \mathrm{Tr}\left[\hat{D}_{-\underline{r}'} \hat{D}_{-\underline{r}} \right]$$

$$= \frac{1}{(2\pi)^{3n}} \iint d\underline{r}\, d\underline{r}' \; e^{i(\underline{r}_2 - \underline{r}_1)^T \Omega \underline{r}} \cdot \delta^{(2n)}(\underline{r}' + \underline{r})$$

$$= \frac{1}{(2\pi)^{3n}} \int d\underline{r}' \; e^{-i(\underline{r}_2 - \underline{r}_1)^T \Omega \underline{r}'}$$

$$= \frac{1}{(2\pi)^{n}} \delta^{(2n)}(\underline{r}_2 - \underline{r}_1)$$

$$\Rightarrow \quad \mathrm{Tr}\left[\hat{A}_{\underline{r}_1} \hat{A}_{\underline{r}_2} \right] = \frac{1}{(2\pi)^n} \delta^{(2n)}(\underline{r}_2 - \underline{r}_1)$$

5) We can also write

$$\rho = (2\pi)^n \int d\underline{r} \; W_\rho(\underline{r}) \, \hat{A}_{\underline{r}}$$

follows b/c

$$\langle \hat{A}_{\underline{r}_1} | \hat{A}_{\underline{r}_2} \rangle \equiv \mathrm{Tr}\left[\hat{A}_{\underline{r}_1} \hat{A}_{\underline{r}_2} \right]$$

$$= \frac{1}{(2\pi)^n} \delta^{(2n)}(\underline{r}_2 - \underline{r}_1)$$

$$\Rightarrow \quad I = (2\pi)^n \int d\underline{r} \; |\hat{A}_{\underline{r}}\rangle\langle \hat{A}_{\underline{r}}|$$

$$\Rightarrow \quad |\rho\rangle = (2\pi)^n \int d\underline{r} \; |\hat{A}_{\underline{r}}\rangle\langle \hat{A}_{\underline{r}} | \rho \rangle$$

$$= (2\pi)^n \int d\underline{r} \; |\hat{A}_r\rangle \, W_\rho(\underline{r})$$

$$\Rightarrow \quad \rho = (2\pi)^n \int d\underline{r} \; W_\rho(\underline{r}) \, \hat{A}_n$$