$\boxed{\text{Lecture 13}}$ 1

Characteristic functions & quasi-probability distributions

recall that for a single mode

$$\hat{D}_{-r} = e^{-ir^T \Omega \hat{r}}$$

& w/ convention

$$\hat{D}_\alpha = e^{\alpha \hat{a}^\dagger - \alpha^* \hat{a}}$$

& $\alpha = \dfrac{x+ip}{\sqrt{2}}$ w/ $r = \begin{bmatrix} x \\ p \end{bmatrix}$

then

$$\hat{D}_{-r} = \hat{D}_\alpha$$

Then define coherent state as

$$\hat{D}_\alpha |0\rangle = |\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

recall that

$$\hat{D}_\alpha \hat{D}_\beta = e^{\frac{1}{2}(\alpha\beta^* - \alpha^*\beta)} \hat{D}_{\alpha+\beta}$$

$$\Rightarrow \quad \langle\beta|\alpha\rangle = \langle 0|\hat{D}_{-\beta} \hat{D}_\alpha|0\rangle$$

$$= \langle 0| \hat{D}_{\alpha-\beta} |0\rangle \, e^{\frac{1}{2}(\alpha\beta^* - \alpha^*\beta)}$$

$$= \langle 0|\alpha-\beta\rangle \, e^{\frac{1}{2}(\alpha\beta^* - \alpha^*\beta)}$$

$$= e^{-\frac{1}{2}|\alpha-\beta|^2} e^{\frac{1}{2}(\alpha\beta^* - \alpha^*\beta)}$$

Coherent states form an overcomplete set:

$$\frac{1}{\pi} \int_{\mathbb{C}} d^2\alpha \, |\alpha\rangle\langle\alpha| = I$$

can then use coherent-state basis to evaluate traces of trace-class operators:

$$\mathrm{Tr}[\hat{O}] = \sum_{m=0}^{\infty} \langle m | \hat{O} | m \rangle$$

$$= \sum_{m=0}^{\infty} \langle m | \frac{1}{\pi} \int_{\mathbb{C}} d^2\alpha \, |\alpha\rangle\langle\alpha| \, \hat{O} \, |m\rangle$$

$$= \frac{1}{\pi} \int_{\mathbb{C}} d^2\alpha \sum_{m=0}^{\infty} \langle m | \alpha \rangle \langle \alpha | \hat{O} | m \rangle$$

$$= \frac{1}{\pi} \int_{\mathbb{C}} d^2\alpha \sum_{m=0}^{\infty} \langle \alpha | \hat{O} | m \rangle \langle m | \alpha \rangle$$

$$= \frac{1}{\pi} \int_{\mathbb{C}} d^2\alpha \, \langle \alpha | \hat{O} | \alpha \rangle$$

can use this to easily figure out mean vector & covariance matrix.

Alternatively, use that (even simpler)

$$\mathrm{Tr}\left[\hat{x}\,|\alpha\rangle\langle\alpha|\right]$$

$$= \langle\alpha|\,\hat{x}\,|\alpha\rangle$$

$$= \langle\alpha|\,\frac{\hat{a}+\hat{a}^{+}}{\sqrt{2}}\,|\alpha\rangle = \frac{\langle\alpha|\left(\hat{a}|\alpha\rangle\right) + \left(\langle\alpha|\hat{a}^{+}\right)|\alpha\rangle}{\sqrt{2}}$$

$$= \frac{\langle\alpha|\,\alpha\,|\alpha\rangle + \langle\alpha|\,\alpha^{*}\,|\alpha\rangle}{\sqrt{2}}$$

$$= \frac{2\,\mathrm{RE}\{\alpha\}}{\sqrt{2}} = \sqrt{2}\,\mathrm{RE}\{\alpha\}$$

Similarly,

$$\langle\alpha|\,\hat{p}\,|\alpha\rangle = \sqrt{2}\,\mathrm{Im}\{\alpha\}$$

Alternatively,

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$$\mathrm{Tr}\left[\hat{x}\,|\alpha\rangle\langle\alpha|\right] =$$

$$\mathrm{Tr}\left\{\hat{x}\,\hat{D}_\alpha\,|0\rangle\langle 0|\,\hat{D}_{-\alpha}\right]$$

$$= \mathrm{Tr}\left\{\hat{D}_{-\alpha}\,\hat{x}\,\hat{D}_\alpha\,|0\rangle\langle 0|\right]$$

$$= \mathrm{Tr}\left[\hat{D}_r\,\hat{x}\,\hat{D}_{-r}\,|0\rangle\langle 0|\right]$$

$$= \mathrm{Tr}\left[(\hat{x} + x)\,|0\rangle\langle 0|\right]$$

$$= \langle 0|\hat{x}|0\rangle + x = x = \sqrt{2}\,\mathrm{RE}\{\alpha\}$$

For covariance matrix,

$$\mathrm{Tr}\left\{\hat{x}^2\,|\alpha\rangle\langle\alpha|\right\}$$

$$= \mathrm{Tr}\left\{\hat{x}^2\,D_\alpha\,|0\rangle\langle 0|\,D_{-\alpha}\right]$$

$$= \mathrm{Tr}\left[D_{\hat{r}}\,\hat{x}^2\,\hat{D}_{-r}\,|0\rangle\langle 0|\right]$$

$$= \mathrm{Tr}\left\{(\hat{x}+x)^2\,|0\rangle\langle 0|\right]$$

$$= \mathrm{Tr}\left[(\hat{x}^2 + 2x\hat{x} + x^2)\,|0\rangle\langle 0|\right]$$

$$= 1 + x^2$$

$$\Rightarrow \quad \mathrm{Tr}\left[(\hat{x} - x)^2 \, |\alpha\rangle\langle\alpha| \right]$$

$$= \underline{\underline{1}}$$

keep calculating + cov. matrix of

$|\alpha\rangle\langle\alpha|$ is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(same as vacuum)

Trace of a displacement operator
in terms of Dirac delta function
(main idea upon which
characteristic functions are
built)

Consider

$$\langle \alpha | D(\beta) | \alpha \rangle$$

$$= \langle 0 | D^{\dagger}(\alpha) D(\beta) D(\alpha) | 0 \rangle$$

$$= \langle 0 | D^{\dagger}(\alpha) \exp\left[\beta \hat{a}^{+} - \beta^{*} \hat{a} \right] D(\alpha) | 0 \rangle$$

$$= \langle 0 | \exp\left[D^{\dagger}(\alpha) \left(\beta \hat{a}^{+} - \beta^{*} \hat{a} \right) D(\alpha) \right] | 0 \rangle$$

$$= \langle 0 | \exp\left[\beta \left(\hat{a}^{+} + \alpha^{*} \right) - \beta^{*} \left(\hat{a} + \alpha \right) \right] | 0 \rangle$$

$$= e^{\beta \alpha^{*} - \beta^{*} \alpha} \langle 0 | \exp\left[\beta \hat{a}^{+} - \beta^{*} \hat{a} \right] | 0 \rangle$$

$$= e^{\beta \alpha^{*} - \beta^{*} \alpha} \langle 0 | D(\beta) | 0 \rangle$$

$$= e^{\beta\alpha^* - \beta^*\alpha} \langle 0|\beta\rangle$$

$$= e^{\beta\alpha^* - \beta^*\alpha} e^{-\frac{1}{2}|\beta|^2}$$

Then $\mathrm{Tr}\left[D(\beta)\right] = \frac{1}{\pi}\int_{\mathbb{C}} d^2\alpha \langle\alpha|D(\beta)|\alpha\rangle$

$$= \frac{1}{\pi} e^{-\frac{1}{2}|\beta|^2} \int_{\mathbb{C}} d^2\alpha \, e^{\beta\alpha^* - \beta^*\alpha}$$

set $\alpha = x + iy$

$\beta = u + iv$

$$\Rightarrow \quad \beta\alpha^* - \beta^*\alpha = (u+iv)(x-iy) - (u-iv)(x+iy)$$

$$= i\left[2vx - 2uy\right]$$

$$\Rightarrow \int_{\mathbb{C}} d^2\alpha \, e^{\beta\alpha^* - \beta^*\alpha} = \iint dx\,dy \, e^{i[2vx - 2uy]}$$

$$= \int dx \, e^{i2vx} \cdot \int dy \, e^{-i2uy}$$

$$= 2\pi\, \delta(2v) \cdot 2\pi\, \delta(2u)$$

$$= \pi\, \delta(v) \cdot \pi\, \delta(u)$$

(used $\delta(ax) = \frac{\delta(x)}{|a|}$)

$$\Rightarrow \mathrm{Tr}\left[D(\beta)\right] = \frac{1}{\pi} e^{-\frac{1}{2}|\beta|^2} \cdot \pi^2\, \delta^2(\beta)$$

$$= \pi\, \delta^2(\beta)$$

since $e^{-\frac{1}{2}|\beta|^2}\Big|_{\beta=0} = 1$

Then by linearity

$$\mathrm{Tr}\left[\hat{D}_\alpha \hat{D}_{-\beta}\right] = \mathrm{Tr}\left[\hat{D}_{\alpha-\beta}\right] \times e^{\frac{1}{2}(-\alpha\beta^* + \alpha^*\beta)}$$

$$= \pi\, \delta^2(\alpha-\beta)\, e^{\frac{1}{2}(-\alpha\beta^* + \alpha^*\beta)}$$

$$= \pi\, \delta^2(\alpha-\beta)$$

since $\delta^2(\alpha-\beta) = 0$ when $\alpha \neq \beta$

& $e^{\frac{1}{2}(-\alpha\beta^* + \alpha^*\beta)} = 1$ when $\alpha = \beta$

$\Rightarrow$ trace orthogonality relationship is

$$\mathrm{Tr}\left[\hat{D}_\alpha \hat{D}_{-\beta}\right] = \pi\, \delta^2(\alpha - \beta)$$

then for $\alpha = \frac{x_r + i p_r}{\sqrt{2}}$ $\beta = \frac{x_s + i p_s}{\sqrt{2}}$

$$\mathrm{Tr}\left[\hat{D}_r \hat{D}_{-s}\right] =$$

$$\pi\, \delta^2(\alpha - \beta)$$

$$= \pi\, \delta\left(\frac{x_r - x_s}{\sqrt{2}}\right) \cdot \delta\left(\frac{p_r - p_s}{\sqrt{2}}\right)$$

$$= 2\pi\, \delta(x_r - x_s) \cdot \delta(p_r - p_s)$$

$$= 2\pi\, \delta^2(r - s)$$

So $\mathrm{Tr}\left[\hat{D}_r \hat{D}_{-s}\right] = 2\pi\, \delta^2(r - s)$

for n modes

$$\mathrm{Tr}\left[\hat{D}_{\underline{\alpha}}\hat{D}_{-\underline{\beta}}\right] = \pi^n \delta^{2n}(\underline{\alpha}-\beta)$$

$$\mathrm{Tr}\left[\hat{D}_{\underline{r}}\ \hat{D}_{-\underline{s}}\right] = (2\pi)^n \delta^{2n}(\underline{r}-\underline{s})$$

Characteristic function for a trace-class operator ρ is defined as

$$\chi_\rho(\alpha) = \mathrm{Tr}[\hat{D}_\alpha \rho]$$

this is finite $\forall \alpha \in \mathbb{C}$ b/c

$$|\mathrm{Tr}[\hat{D}_\alpha \rho]| \le \|\hat{D}_\alpha\|_\infty \|\rho\|_1$$
$$= \|\rho\|_1 < \infty$$

Hölder inequality.

can then write

$$\rho = \int d^2\alpha \, \chi_\rho(\alpha) \, \hat{D}_{-\alpha}$$

~~Consider that~~

Proof: $\cancel{\mathrm{Tr}[\hat{D}_\beta \rho] =}$

Since $\rho = \left[\frac{1}{\pi}\int d^2\alpha \, |\alpha\rangle\langle\alpha|\right] \rho \left[\frac{1}{\pi}\int d^2\beta \, |\beta\rangle\langle\beta|\right]$

$$= \frac{1}{\pi^2} \iint d^2\alpha \, \langle\alpha|\rho|\beta\rangle \, |\alpha\rangle\langle\beta|$$

it suffices to show that

$$|\alpha\rangle\langle\beta| = \frac{1}{\pi}\int d^2\gamma \, \mathrm{Tr}\left[|\alpha\rangle\langle\beta| \hat{D}_\gamma\right] \hat{D}_{-\gamma}$$

This is the same as

$$|0\rangle\langle 0| = \frac{1}{\pi}\int d^2\gamma \, \mathrm{Tr}\left[|\alpha\rangle\langle\beta| \hat{D}_\gamma\right] \hat{D}_{-\alpha}\hat{D}_{-\gamma}\hat{D}_\beta$$

$$= \frac{1}{\pi}\int d^2\gamma \, \langle\beta-\gamma|\alpha\rangle \, e^{\frac{1}{2}(\gamma\beta^* - \gamma^*\beta)} \, \hat{D}_{-\alpha}\hat{D}_{-\gamma}\hat{D}_\beta$$

$$= \frac{1}{\pi} \int d^2\gamma \; e^{-\frac{1}{2}|\beta-\alpha-\gamma|^2} \; \hat{D}_{\beta-\alpha-\gamma}$$

$$= \frac{1}{\pi} \int d^2\gamma \; e^{-\frac{1}{2}|\gamma|^2} \; \hat{D}_{\gamma}$$

pick up next time.