Lecture 12

In quantum information, we are often interested in computing overlap formulas of the following kind:

$$\mathrm{Tr}[\rho\sigma]$$

$$F_H(\rho,\sigma) = \mathrm{Tr}[\sqrt{\rho}\sqrt{\sigma}]^2 \quad \text{(Holevo fidelity - 1972)}$$

$$F(\rho,\sigma) = \|\sqrt{\rho}\sqrt{\sigma}\|_1^2 = \mathrm{Tr}\left[\sqrt{\sqrt{\sigma}\rho\sqrt{\sigma}}\right]^2$$

(Uhlmann fidelity) - 1976

& more generally Renyi overlap formulas of the following kind:

$$Q_\alpha(\rho,\sigma) = \mathrm{Tr}[\rho^\alpha \sigma^{1-\alpha}] \quad \text{for } \alpha \in (0,1) \cup (1,\infty)$$

&

$$\tilde{Q}_\alpha(\rho,\sigma) = \mathrm{Tr}\left[\left(\sigma^{\frac{1-\alpha}{2\alpha}} \rho\, \sigma^{\frac{1-\alpha}{2\alpha}}\right)^\alpha\right] \quad \text{for } \alpha \in (0,1) \cup (1,\infty)$$

Note that $Q_{\alpha=1/2}(\rho,\sigma) = \sqrt{F_H(\rho,\sigma)}$

$$\tilde{Q}_{\alpha=1/2}(\rho,\sigma) = \sqrt{F}(\rho,\sigma)$$

these functions are interesting because $F_H(\rho,\sigma) \leq F(\rho,\sigma)$ &

$$1-\sqrt{F(\rho,\sigma)} \leq 1-\sqrt{F_H(\rho,\sigma)} \leq \frac{1}{2}\|\rho-\sigma\|_1 \leq \sqrt{1-F(\rho,\sigma)} \leq \sqrt{1-F_H(\rho,\sigma)}$$

both related to operationally meaningful trace distance, for which we don't have a general formula for Gaussian states, but we can use the above to bound it.

We will only consider the ~~zero~~-mean case for simplicity.

Let us first consider the simple overlap formula:

$$\mathrm{Tr}[\rho\, \tau]$$

for
$$\rho = \frac{1}{\sqrt{\mathrm{Det}\left(\frac{\sigma_\rho + i\Omega}{2}\right)}} \exp\left(-\frac{1}{2}\hat{r}^T H_\rho \hat{r}\right)$$

$$\tau = \frac{1}{\sqrt{\mathrm{Det}\left(\frac{\sigma_\tau + i\Omega}{2}\right)}} \exp\left(-\frac{1}{2}\hat{r}^T H_\tau \hat{r}\right)$$

So it is clear that

$$\mathrm{Tr}[\rho\,\tau] = \frac{1}{\sqrt{\mathrm{Det}\left(\frac{\sigma_\rho + i\Omega}{2}\right)\mathrm{Det}\left(\frac{\sigma_\tau + i\Omega}{2}\right)}} \mathrm{Tr}\left[e^{-\frac{1}{2}\hat{r}^T H_\rho \hat{r}}\, e^{-\frac{1}{2}\hat{r}^T H_\tau \hat{r}}\right]$$

From homework, recall that for complex symmetric H_1 + H_2,
$\exists$ " " H_3 such that H_3

satisfying $e^{-\frac{1}{2}\hat{r}^T H_1 \hat{r}} e^{-\frac{1}{2}\hat{r}^T H_2 \hat{r}} = e^{-\frac{1}{2}\hat{r}^T H_3 \hat{r}}$

also satisfies $e^{-i\Lambda H_1} e^{-i\Lambda H_2} = e^{-i\Lambda H_3}$

$$\Rightarrow \quad e^{i\Lambda H_3} = e^{i\Lambda H_2} e^{i\Lambda H_1}$$

by taking inverses

then define $W_3 = (I + e^{i\Lambda H_3})(I - e^{i\Lambda H_3})^{-1}$

& $\sigma_3 = -W_3 i\Lambda$

(i.e., $\sigma_3 = \coth(i\Lambda H_3/2)\, i\Lambda$)

Also, $\sigma_1 = \coth(i\Lambda H_1/2)\, i\Lambda$

$\sigma_2 = \coth(i\Lambda H_2/2)\, i\Lambda$

After a long sequence of algebraic steps, we find that (see page 13 of Mob. 07885)

$$\sigma_3 = -i\Lambda + (\sigma_2 + i\Lambda)(\sigma_1 + \sigma_2)^{-1}(\sigma_1 + i\Lambda)$$

$$H_3 = 2i\Lambda \operatorname{arccoth}(\sigma_3 i\Lambda)$$

Note that σ_3 is complex symmetric

Observe that

$$\sqrt{\mathrm{Det}\left(\frac{\sigma_3 + i\Lambda}{2}\right)}$$

$$= \sqrt{\mathrm{Det}\left(\left(\frac{\sigma_2 + i\Lambda}{2}\right)\left(\frac{\sigma_1 + \sigma_2}{2}\right)^{-1}\left(\frac{\sigma_1 + i\Lambda}{2}\right)\right)}$$

So if indeed $\mathrm{Tr}\left[e^{-\frac{1}{2}\hat{r}^T H_3 \hat{r}}\right]$

$$= \sqrt{\mathrm{Det}\left(\frac{\sigma_3 + i\Lambda}{2}\right)}$$

Then applying the above gives

$$\mathrm{Tr}[\rho\,\tau] =$$

$$\frac{1}{\sqrt{\mathrm{Det}\left(\frac{\sigma_\rho + i\Omega}{2}\right)\mathrm{Det}\left(\frac{\sigma_\tau + i\Omega}{2}\right)}} \times \sqrt{\frac{\mathrm{Det}\left(\frac{\sigma_\rho + i\Omega}{2}\right)\mathrm{Det}\left(\frac{\sigma_\tau + i\Omega}{2}\right)}{\mathrm{Det}\left(\frac{\sigma_\rho + \sigma_\tau}{2}\right)}}$$

$$= \frac{1}{\sqrt{\mathrm{Det}\left(\frac{\sigma_\rho + \sigma_\tau}{2}\right)}} = \frac{2^n}{\sqrt{\mathrm{Det}(\sigma_\rho + \sigma_\tau)}}$$

which is in fact the correct overlap formula.

It can in fact be shown that

$$\mathrm{Tr}\left[e^{-\frac{1}{2}\hat{r}^T H_3 \hat{r}}\right] = \sqrt{\mathrm{Det}\left(\frac{\sigma_3 + i\Omega}{2}\right)}$$

for H_1 + H_2 given.

See Prop. 11 of 1706.09885

Now let us compute a formula for the Petz-Renyi overlap

$$Q_\alpha(\rho, \tau) = \mathrm{Tr}\left[\rho^{\alpha} \tau^{1-\alpha}\right] \qquad 1^{st} \text{ for } \alpha \in (0,1)$$

for $$\rho = \frac{\exp\left(-\frac{1}{2}\hat{r}^T H_\rho \hat{r}\right)}{\sqrt{\mathrm{Det}\left(\frac{\sigma_\rho + i\Omega}{2}\right)} = Z_\rho}$$

$$\tau = \frac{\exp\left(-\frac{1}{2}\hat{r}^T H_\tau \hat{r}\right)}{\sqrt{\mathrm{Det}\left(\frac{\sigma_\tau + i\Omega}{2}\right)} = Z_\tau}$$

$$\Rightarrow Q_\alpha(\rho, \tau) = \frac{1}{Z_\rho^{\alpha} Z_\tau^{1-\alpha}} \mathrm{Tr}\left[e^{-\frac{1}{2}\hat{r}^T(\alpha H_\rho)\hat{r}} e^{-\frac{1}{2}\hat{r}^T[(1-\alpha) H_\tau]\hat{r}}\right]$$

$$= \frac{Z_{\rho(\alpha)} Z_{\tau(1-\alpha)}}{Z_\rho^{\alpha} Z_\tau^{1-\alpha}} \mathrm{Tr}\left[\frac{e^{-\frac{1}{2}\hat{r}(\alpha H_\rho)\hat{r}}}{Z_{\rho(\alpha)}} \frac{e^{-\frac{1}{2}(\hat{r}^T(1-\alpha)H_\tau \hat{r})}}{Z_{\tau(1-\alpha)}}\right]$$

Now use overlap formula for Gaussian states to get

$$= \frac{Z_{\rho(\alpha)} \, Z_{\tau(1-\alpha)}}{Z_\rho^{\alpha} \, Z_\tau^{1-\alpha} \sqrt{\mathrm{Det}\left(\frac{\sigma_{\rho(\alpha)} + \sigma_{\tau(1-\alpha)}}{2}\right)}}$$

where

$$Z_{\rho(\alpha)} = \sqrt{\mathrm{Det}\left(\frac{\sigma_{\rho(\alpha)} + i\Lambda}{2}\right)}$$

$$Z_{\tau(1-\alpha)} = \sqrt{\mathrm{Det}\left(\frac{\sigma_{\tau(1-\alpha)} + i\Lambda}{2}\right)}$$

$$\sigma_{\rho(\alpha)} = \frac{\left[I + (\sigma_\rho i\Lambda)^{-1}\right]^{\alpha} + \left[I - (\sigma_\rho i\Lambda)^{-1}\right]^{\alpha}}{\left[I + (\sigma_\rho i\Lambda)^{-1}\right]^{\alpha} - \left[I - (\sigma_\rho i\Lambda)^{-1}\right]^{\alpha}} \times i\Lambda$$

$$\sigma_{\tau(1-\alpha)} = \frac{\left[I + (\sigma_\tau i\Lambda)^{-1}\right]^{1-\alpha} + \left[I - (\sigma_\tau i\Lambda)^{-1}\right]^{1-\alpha}}{\left[I + (\sigma_\tau i\Lambda)^{-1}\right]^{1-\alpha} - \left[I - (\sigma_\tau i\Lambda)^{-1}\right]^{1-\alpha}} \times i\Lambda$$

For $\alpha = 1/2$, this simplifies to

$$Q_{1/2}(\rho,\tau) = \mathrm{Tr}\left[\sqrt{\rho}\sqrt{\tau}\right]$$

$$= \frac{Z_{\rho(1/2)}\, Z_{\tau(1/2)}}{Z_\rho^{1/2}\, Z_\tau^{1/2}\sqrt{\mathrm{Det}\left(\frac{\sigma_{\rho(1/2)} + \sigma_{\tau(1/2)}}{2}\right)}}$$

where $\sigma_{\rho(1/2)} = \left(\sqrt{I + (\sigma_\rho \Omega)^{-2}} + I\right)\sigma_\rho$

$$\sigma_{\tau(1/2)} = \left(\sqrt{I + (\sigma_\tau \Omega)^{-2}} + I\right)\sigma_\tau$$

$$Z_{\rho(1/2)} = \sqrt{\mathrm{Det}\left(\frac{\sigma_{\rho(1/2)} + i\Omega}{2}\right)}$$

$$Z_{\tau(1/2)} = \sqrt{\mathrm{Det}\left(\frac{\sigma_{\tau(1/2)} + i\Omega}{2}\right)}$$

What about when $\alpha > 1$?

It can be shown for $H_1 > 0$ & $H_2 > 0$ such that $\sigma_2 > \sigma_1$

$$\mathrm{Tr}\left[e^{-\frac{1}{2}\hat{r}^T H_1 \hat{r}} \, e^{-\frac{1}{2}\hat{r}^T (-H_2)\hat{r}} \right]$$

$$= \sqrt{\frac{\mathrm{Det}\left(\frac{\sigma_1 + i\Omega}{2}\right) \mathrm{Det}\left(\frac{\sigma_2 + i\Omega}{2}\right)}{\mathrm{Det}\left(\frac{\sigma_2 - \sigma_1}{2}\right)}}$$

Then consider

$$\mathrm{Tr}\left[\rho^{\alpha} \tau^{1-\alpha}\right]$$

$$= \frac{1}{Z_\rho^{\alpha} Z_\tau^{1-\alpha}} \mathrm{Tr}\left[e^{-\frac{1}{2}\hat{r}^T(\alpha H_\rho)\hat{r}} \times e^{-\frac{1}{2}\hat{r}^T[-(\alpha-1) H_\tau]\hat{r}} \right]$$

Apply above to get

$$= \frac{Z_{\rho(\alpha)}\, Z_{\tau(\alpha-1)}}{Z_\rho^{\alpha}\, Z_\tau^{1-\alpha}} \frac{1}{\sqrt{\mathrm{Det}\left(\frac{\sigma_{\tau(\alpha-1)} - \sigma_{\rho(\alpha)}}{2}\right)}}$$

for $\alpha \geq 1$

where $Z_{\rho(\alpha)}$ $Z_{\tau(\alpha-1)}$

$\sigma_{\rho(\alpha)}$ & $\sigma_{\tau(\alpha-1)}$ are defined similarly.

Interesting special case for $\alpha = 2$

$\mathrm{Tr}[\rho^2 \tau^{-1}]$

$$= \frac{Z_\tau^2\, Z_{\rho(2)}}{Z_\rho^2 \sqrt{\mathrm{Det}\left(\frac{\sigma_\tau - \sigma_{\rho(2)}}{2}\right)}}$$

where $\sigma_{\rho(2)} = \frac{1}{2}(\sigma_\rho + \Omega \sigma_\rho^{-1} \Omega^T)$

Now what about

$$\tilde{Q}_\alpha(\rho,\tau) = \mathrm{Tr}\left[\left(\tau^{\frac{1-\alpha}{2\alpha}}\,\rho\,\tau^{\frac{1-\alpha}{2\alpha}}\right)^{\alpha}\right] \quad ?$$

First, use that

$$\tilde{Q}_\alpha(\rho,\tau) = \mathrm{Tr}\left[\left(\rho^{1/2}\,\tau^{\frac{1-\alpha}{\alpha}}\,\rho^{1/2}\right)^{\alpha}\right]$$

to simplify

Now, with

$$\rho = \frac{1}{\underbrace{\sqrt{\mathrm{Det}\left(\frac{\sigma_\rho + i\Lambda}{2}\right)}}_{=Z_\rho}} \exp\left(-\frac{1}{2}\hat{r}^T H_\rho \hat{r}\right)$$

$$\tau = \frac{1}{\underbrace{\sqrt{\mathrm{Det}\left(\frac{\sigma_\tau + i\Lambda}{2}\right)}}_{=Z_\tau}} \exp\left(-\frac{1}{2}\hat{r}^T H_\tau \hat{r}\right)$$

Define $\beta = \frac{1-\alpha}{\alpha}$

$$\Rightarrow \quad \tilde{Q}_\alpha(\rho,\tau) = \frac{1}{Z_\rho^\alpha Z_\tau^{1-\alpha}} \operatorname{Tr}\left[\left[e^{-\frac{1}{2}\hat{r}^T (1/2 H_\rho)\hat{r}} e^{-\frac{1}{2}\hat{r}^T \beta H_\tau \hat{r}} \cdot e^{-\frac{1}{2}\hat{r}^T (1/2 H_\rho)\hat{r}} \right]^\alpha \right]$$

use that

$$e^{-\frac{1}{2}\hat{r}^T (H_1/2)\hat{r}} e^{-\frac{1}{2}\hat{r}^T H_2 \hat{r}} e^{-\frac{1}{2}\hat{r}^T (H_1/2)\hat{r}}$$

$$= e^{-\frac{1}{2}\hat{r}^T H_3 \hat{r}}$$

where $H_3 = 2i\Omega \operatorname{arcoth}(\sigma_3 i\Omega)$

&

$$\sigma_3 = \sigma_1 - \left(\sqrt{I + (\sigma_1\Omega)^{-2}}\right) \sigma_1 (\sigma_1 + \sigma_2)^{-1} \sigma_1 \times \sqrt{I + (\Omega\sigma_1)^{-2}}$$

See Prop. 8 of 1706.09885

Applying this, we find that

$$e^{-\frac{1}{2}\hat{r}^T(\frac{1}{2}H_\rho)\hat{r}} e^{-\frac{1}{2}\hat{r}^T \beta H_\tau \hat{r}} e^{-\frac{1}{2}\hat{r}^T(\frac{1}{2}H_\rho)\hat{r}}$$

$$= e^{-\frac{1}{2}\hat{r}^T H_\xi \hat{r}}$$

where $H_\xi = 2i\Omega \,\mathrm{arcoth}(\sigma_\xi i\Omega)$

$$\sigma_\xi = \sigma_\rho - \sqrt{I+(\sigma_\rho\Omega)^{-2}}\,\sigma_\rho(\sigma_\rho+\sigma_{\tau(\beta)})^{-1}\sigma_\rho\sqrt{I+(\Omega\sigma_\rho)^{-2}}$$

$$\sigma_{\tau(\beta)} = \frac{[I+(\sigma_\tau i\Omega)^{-1}]^\beta + [I-(\sigma_\tau i\Omega)^{-1}]^\beta}{[I+(\sigma_\tau i\Omega)^{-1}]^\beta - [I-(\sigma_\tau i\Omega)^{-1}]^\beta} i\Omega$$

Then we take power α to get

$$e^{-\frac{1}{2}\hat{r}^T \alpha H_\xi \hat{r}}$$

&

$$\mathrm{Tr}\left\{e^{-\frac{1}{2}\hat{r}^T \alpha H_\xi \hat{r}}\right\} = \sqrt{\mathrm{Det}\left(\frac{\sigma_{\xi(\alpha)}+i\Omega}{2}\right)}$$

where

$$\sigma_{\xi(\alpha)} = \frac{[I+(\sigma_\xi i\Omega)^{-1}]^\alpha + [I-(\sigma_\xi i\Omega)^{-1}]^\alpha}{[I+(\sigma_\xi i\Omega)^{-1}]^\alpha - [I-(\sigma_\xi i\Omega)^{-1}]^\alpha}$$

$\Rightarrow$ $\tilde{Q}_\alpha(\rho,\tau)$

$$= \frac{1}{Z_\rho^\alpha Z_\tau^{1-\alpha}} \sqrt{\mathrm{Det}\left(\frac{\sigma_{\xi(\alpha)} + i\Omega}{2}\right)}$$

so gives formula for sandwiched Renyi for $\alpha \in (0,1)$

What about case of fidelity?

$$\tilde{Q}_{\alpha=1/2}(\rho,\tau) = \mathrm{Tr}\left[\sqrt{\rho^{1/2}\tau\rho^{1/2}}\right]$$

if $\alpha = 1/2$ then $\beta = \frac{1-\alpha}{\alpha} = 1$

$\Rightarrow \tau(\beta) = \tau$

$$\Rightarrow \sigma_\xi = \sigma_\rho - \sqrt{I + (\sigma_\rho\Omega)^{-2}}\,\sigma_\rho(\sigma_\rho + \sigma_\tau)^{-1}\sigma_\rho \times \sqrt{I + (\Omega\sigma_\rho)^{-2}}$$

$$\& \quad \sigma_{\xi(\alpha=1/2)} = \left(\sqrt{I + (\sigma_\xi\Omega)^{-2}} + I\right)\sigma_\xi$$

& then fidelity becomes

$$\mathrm{Tr}\left[\sqrt{\rho^{1/2}\tau\rho^{1/2}}\right] = \frac{\sqrt{\mathrm{Det}\left(\frac{\sigma_{\xi(1/2)}+i\Lambda}{2}\right)}}{Z_\rho^{1/2} Z_\tau^{1/2}}$$

Sandwiched Renyi for $\alpha > 1$ evaluates to when $\sigma_{\tau(\gamma)} > \sigma_\rho$

$$\tilde{Q}_\alpha(\rho,\tau) = \frac{1}{Z_\rho^\alpha Z_\tau^{1-\alpha}}\sqrt{\mathrm{Det}\left(\frac{\sigma_{\xi(\alpha)}+i\Lambda}{2}\right)}$$

$$\sigma_{\xi(\alpha)} = \frac{\left[I+(\sigma_\xi i\Lambda)^{-1}\right]^\alpha + \left[I-(\sigma_\xi i\Lambda)^{-1}\right]^\alpha}{\left[I+(\sigma_\xi i\Lambda)^{-1}\right]^\alpha - \left[I-(\sigma_\xi i\Lambda)^{-1}\right]^\alpha} \times i\Lambda$$

$$\sigma_\xi = \sigma_\rho + \sqrt{I+(\sigma_\rho\Lambda)^{-2}}\,\sigma_\rho\left(\sigma_{\tau(\gamma)} - \sigma_\rho\right)^{-1}\sigma_\rho \times \sqrt{I+(\Lambda\sigma_\rho)^{-2}}$$

$$\sigma_{\tau(\gamma)} = \frac{\left[I + (\sigma_\tau i\Lambda)^{-1}\right]^\gamma + \left[I - (\sigma_\tau i\Lambda)^{-1}\right]^\gamma}{\left[I + (\sigma_\tau i\Lambda)^{-1}\right]^\gamma - \left[I - (\sigma_\tau i\Lambda)^{-1}\right]^\gamma} \cdot i\Lambda$$

for $\gamma = \frac{\alpha - 1}{\alpha}$