Lecture 11

Computing relative entropy of faithful Gaussian states

$$D(\rho\|\tau) = \mathrm{Tr}\{\rho[\log\rho - \log\tau]\}$$

We already showed that

$$\mathrm{Tr}[\rho\log\rho] =$$

$$-\frac{1}{2}\log \mathrm{Det}\left(\frac{\sigma_\rho + i\Omega}{2}\right) - \frac{1}{4}\mathrm{Tr}[H_\rho\sigma_\rho]$$

Now what about

$$-\mathrm{Tr}[\rho\log\tau] \qquad ?$$

Consider that

$$\rho = \hat{D}_{-\bar{r}_\rho}\,\rho_0\,\hat{D}_{\bar{r}_\rho}$$

where ρ_0 has zero mean & same covariance matrix σ_ρ as ρ.

Then using cyclicity of trace
& functional calculus of log,
we find that

$$-\mathrm{Tr}\left[\rho \log \tau\right]$$

$$= -\mathrm{Tr}\left[\rho_0 \log \hat{D}_{\bar{r}_\rho}\, \tau\, \hat{D}_{-\bar{r}_\rho}\right]$$

Now consider that

$$\tau = \frac{e^{-\frac{1}{2}(\hat{r}-\bar{r}_\tau)^T H_\tau (\hat{r}-\bar{r}_\tau)}}{\sqrt{\mathrm{Det}\left(\frac{\sigma_\tau + i\Omega}{2}\right)}}$$

so that

$$\hat{D}_{\bar{r}_\rho}\, \tau\, \hat{D}_{-\bar{r}_\rho} = \frac{e^{-\frac{1}{2}(\hat{r}-\delta)^T H_\tau (\hat{r}-\delta)}}{\sqrt{\mathrm{Det}\left(\frac{\sigma_\tau + i\Omega}{2}\right)}}$$

w/ $\delta \equiv \bar{r}_\tau - \bar{r}_\rho$

$$\Rightarrow -\mathrm{Tr}\left[\rho_0 \log \hat{D}_{\bar{r}_\rho}\, t\, \hat{D}_{-\bar{r}_\rho}\right]$$

$$= -\mathrm{Tr}\left[\rho_0 \log \frac{1}{\sqrt{\mathrm{Det}\left(\frac{\sigma_\tau + i\Omega}{2}\right)}}\right]$$

$$+ \mathrm{Tr}\left[\rho_0 \left(\tfrac{1}{2}(\hat{r}-\delta)^T H_\tau (\hat{r}-\delta)\right)\right]$$

$$= \frac{1}{2} \log \operatorname{Det}\left(\frac{\sigma_\tau + i\Omega}{2}\right)$$

$$+ \frac{1}{2} \operatorname{Tr}\left[\rho_0 (\hat{r} - \delta)^T H_\tau (\hat{r} - \delta)\right]$$

Focus on 2^{nd} term & it is equal to

$$\frac{1}{2} \sum_{jk} \operatorname{Tr}\left[\rho_0 (\hat{r}_j - \delta_j)(\hat{r}_k - \delta_k)\right] H^\tau_{jk}$$

$$= \frac{1}{2} \sum_{jk} \Big(\operatorname{Tr}\left[\rho_0 \hat{r}_j \hat{r}_k\right] - \operatorname{Tr}\left[\rho_0 \hat{r}_k\right] \delta_j - \operatorname{Tr}\left[\rho_0 \hat{r}_j\right] \delta_k + \operatorname{Tr}\left[\rho_0\right] \delta_j \delta_k \Big) H^\tau_{jk}$$

$$= \frac{1}{2} \sum_{jk} \left(\operatorname{Tr}\left[\rho_0 \hat{r}_j \hat{r}_k\right] + \delta_j \delta_k \right) H^\tau_{jk}$$

$$= \frac{1}{2} \left[\sum_{jk} \operatorname{Tr}\left[\rho_0 \hat{r}_j \hat{r}_k\right] H^\tau_{jk} \right] + \frac{1}{2} \delta^T H_\tau \delta$$

we calculated this previously to be $\frac{1}{4} \operatorname{Tr}\left[\sigma_\rho H_\tau\right]$

$\Rightarrow$ formula for ~~relative entropy~~ is

$$-\mathrm{Tr}[\rho \log \tau]$$

$$\rightarrow -\mathrm{Tr}[\rho \log \tau] = \frac{1}{2} \log \mathrm{Det}\left(\frac{\sigma_\tau + i\Omega}{2}\right) + \frac{1}{4}\mathrm{Tr}[\sigma_\rho H_\tau] + \frac{1}{2}\delta^T H_\tau \delta$$

w/ $\delta = \bar{r}_\tau - \bar{r}_\rho$

$$\Rightarrow \boxed{\begin{aligned} D(\rho\|\tau) &= \frac{1}{2}\log\left(\frac{\mathrm{Det}\left(\frac{\sigma_\tau + i\Omega}{2}\right)}{\mathrm{Det}\left(\frac{\sigma_\rho + i\Omega}{2}\right)}\right) \\ &+ \frac{1}{4}\mathrm{Tr}[\sigma_\rho (H_\tau - H_\rho)] \\ &+ \frac{1}{2}\delta^T H_\tau \delta \end{aligned}}$$

Computing Renyi entropies & powers of Gaussian states

Interested in

$$\mathrm{Tr}[\rho^\alpha] \qquad \text{for } \alpha \in (0,1) \cup (1,\infty)$$

Using the fact that

$$\rho = \hat{D}_{-\bar{r}} \hat{S} \left(\bigotimes_{j=1}^{n} \theta(\bar{n}_j) \right) \hat{S}^\dagger \hat{D}_{\bar{r}}$$

we find that it reduces to

$$\mathrm{Tr}[\rho^\alpha] = \prod_{j=1}^{n} \mathrm{Tr}[\theta(\bar{n}_j)^\alpha]$$

So compute

$$\mathrm{Tr}\{\theta(\bar{n})^\alpha\} = \frac{1}{(\bar{n}+1)^\alpha} \sum_{n=0}^{\infty} \left(\frac{\bar{n}}{\bar{n}+1}\right)^{\alpha n}$$

$$= \frac{1}{(\bar{n}+1)^\alpha} \frac{1}{1 - \left(\frac{\bar{n}}{\bar{n}+1}\right)^\alpha}$$

$$= \frac{1}{(\bar{n}+1)^\alpha - \bar{n}^\alpha}$$

$$\Rightarrow \mathrm{Tr}[\rho^\alpha] = \prod_{j=1}^{n} \frac{1}{(\bar{n}_j+1)^\alpha - \bar{n}_j^\alpha}$$

In terms of symp. eigenval's

$$r_j = 2\bar{n}_j + 1$$

$$\Rightarrow \frac{r_j - 1}{2} = \bar{n}_j \qquad \frac{r_j+1}{2} = \bar{n}_j + 1$$

$$\Rightarrow \mathrm{Tr}[\rho^\alpha] = \prod_{j=1}^{n} \frac{2^\alpha}{(r_j+1)^\alpha - (r_j-1)^\alpha}$$

Renyi entropy is

$$S_\alpha(\rho) = \frac{1}{1-\alpha} \log \mathrm{Tr}[\rho^\alpha]$$

$$\Rightarrow S_\alpha(\rho) = \frac{1}{1-\alpha} \sum_{j=1}^{n} \log\left(\frac{2^\alpha}{(r_j+1)^\alpha - (r_j-1)^\alpha}\right)$$

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Can also write Renyi entropy as

$$S_\alpha(\rho) = \frac{\alpha}{1-\alpha} \log \mathrm{Tr}[\rho^\alpha]^{1/\alpha}$$

$$= \frac{\alpha}{1-\alpha} \log \|\rho\|_\alpha$$

$$\Rightarrow \quad S_\infty(\rho) = -\log \underbrace{\|\rho\|_\infty}_{\text{largest eigenvalue}} \equiv S_{min}(\rho)$$

Use that $\quad \|\theta(\bar{n})\|_\infty = \frac{1}{\bar{n}+1}$

$$\Rightarrow \quad S_\infty(\rho) = -\log \|\bigotimes_{j=1}^{n} \theta(\bar{n}_j)\|_\infty$$

$$= -\log \prod_{j=1}^{n} \|\theta(\bar{n}_j)\|_\infty$$

$$= \sum_{j=1}^{n} -\log\left(\frac{1}{\bar{n}_j+1}\right)$$

$$= \sum_{j=1}^{n} \log(\bar{n}_j+1)$$

$$= \sum_{j=1}^{n} \log\left(\frac{\nu_j+1}{2}\right)$$

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In general, we have

$$S_\alpha(\rho) \geq S_\beta(\rho) \quad \text{for} \quad \alpha \leq \beta$$

formulas above imply that the gap

$$S(\rho) - S_\infty(\rho)$$

$$= \sum_{j=1}^{n} g(\bar{n}_j) - \log(\bar{n}_j+1)$$

$$= \sum_{j=1}^{n} (\bar{n}_j+1)\log(\bar{n}_j+1) - \bar{n}_j \log n_j - \log(\bar{n}_j+1)$$

$$= \sum_{j=1}^{n} \log\left(\left[\frac{\bar{n}_j+1}{\bar{n}_j}\right]^{\bar{n}_j}\right)$$

$$\leq \sum_{j=1}^{n} \log(e) = n$$

so the gap never exceeds the number of modes.

What about power of Gaussian state?

$$\rho^{\alpha} \quad + \quad \frac{\rho^{\alpha}}{\mathrm{Tr}\{\rho^{\alpha}\}}$$

1st focus on 2nd one $\frac{\rho^{\alpha}}{\mathrm{Tr}\{\rho^{\alpha}\}}$

For a faithful Gaussian state, we have

$$\rho = \frac{\exp\left(-\frac{1}{2}(\hat{r}-\bar{r})^{T} H (\hat{r}-\bar{r})\right)}{\sqrt{\mathrm{Det}\left(\frac{\sigma+i\Omega}{2}\right)}}$$

$$= \frac{\hat{D}_{-\bar{r}} \exp\left[-\frac{1}{2}\hat{r}^{T} H \hat{r}\right] \hat{D}_{\bar{r}}}{\sqrt{\mathrm{Det}\left(\frac{\sigma+i\Omega}{2}\right)}}$$

$$\Rightarrow \rho^{\alpha} \propto \hat{D}_{-\bar{r}} \exp\left[-\frac{1}{2}\hat{r}^{T} \alpha H \hat{r}\right] \hat{D}_{r}$$

Defining $H_{(\alpha)} \equiv \alpha H$

then there is a corresponding

$\sigma_{(\alpha)}$ such that normalized

$$\frac{\rho^{\alpha}}{\mathrm{Tr}[\rho^{\alpha}]} = \frac{\hat{D}_{-\bar{r}} \exp\left[-\frac{1}{2}\hat{r}^{T} H_{(\alpha)} \hat{r}\right] \hat{D}_{\bar{r}}}{\sqrt{\mathrm{Det}\left(\frac{\sigma_{(\alpha)} + i\Omega}{2}\right)}}$$

$$= \frac{\exp\left[-\frac{1}{2}(\hat{r}-\bar{r})^{T} H_{(\alpha)} (\hat{r}-\bar{r})\right]}{\sqrt{\mathrm{Det}\left(\frac{\sigma_{(\alpha)} + i\Omega}{2}\right)}}$$

use the formulas

$$\sigma = \coth(i\Omega H/2)\, i\Omega$$

to write $H = 2i\Omega \,\mathrm{arcoth}(\sigma i\Omega)$,

$$\sigma_{(\alpha)} = \coth(i\Omega H_{(\alpha)}/2)\, i\Omega$$

$$= \coth(i\Omega \alpha H/2)\, i\Omega$$

$$= \coth\left(i\Omega \frac{\alpha}{2}\left[2i\Omega \,\mathrm{arcoth}(\sigma i\Omega)\right]\right) i\Omega$$

$$= \coth(\alpha \,\mathrm{arcoth}(\sigma i\Omega))\, i\Omega$$

Now consider for $|x| > 1$ that

$$\coth(\alpha \operatorname{arcoth}(x)) = \frac{(1+1/x)^\alpha + (1-1/x)^\alpha}{(1+1/x)^\alpha - (1-1/x)^\alpha}$$

$\sigma i\Omega$ has all eigenvalues >1 or <-1, so that this applies $\forall$

$$\boxed{\sigma_{(\alpha)} = \left(\frac{[I + (\sigma i\Omega)^{-1}]^\alpha + [I - (\sigma i\Omega)^{-1}]^\alpha}{[I + (\sigma i\Omega)^{-1}]^\alpha - [I - (\sigma i\Omega)^{-1}]^\alpha} \right) \times i\Omega}$$

$$\Rightarrow \frac{\rho^\alpha}{Tr[\rho^\alpha]} = \frac{\exp\left[-\frac{1}{2}(\hat{r}-\bar{r})^T H_{(\alpha)} (\hat{r}-\bar{r})\right]}{\sqrt{Det\left(\frac{\sigma_{(\alpha)} + i\Omega}{2}\right)}}$$

$$\Rightarrow \mathrm{Tr}\{\rho^\alpha\} = \mathrm{Tr}\left[\left(\frac{\exp\left[-\frac{1}{2}(\hat{r}-\bar{r})^T H (\hat{r}-\bar{r})\right]}{\sqrt{\mathrm{Det}\left(\frac{\sigma + i\Omega}{2}\right)}}\right)^\alpha\right]$$

$$= \frac{1}{\left[\mathrm{Det}\left(\frac{\sigma+i\Omega}{2}\right)\right]^{\alpha/2}} \mathrm{Tr}\left[\exp\left(-\frac{1}{2}(\hat{r}-\bar{r})^T \alpha H (\hat{r}-\bar{r})\right)\right]$$

$$= \frac{1}{\left[\mathrm{Det}\left(\frac{\sigma+i\Omega}{2}\right)\right]^{\alpha/2}} \sqrt{\mathrm{Det}\left(\frac{\sigma(\alpha)+i\Omega}{2}\right)}$$

Interesting special cases include

$\alpha = 2$ & $\alpha = 1/2$

for $\alpha = 2$, we have

$$\frac{(1+1/x)^\alpha + (1-1/x)^\alpha}{(1+1/x)^\alpha - (1-1/x)^\alpha} = \frac{1}{2}(x + x^{-1})$$

$$\Rightarrow \sigma_{(2)} = \frac{1}{2}\left[\sigma i\Omega + (\sigma i \Omega)^{-1}\right] i\Omega$$

$$= \frac{1}{2}\left[\sigma + i\Omega\sigma^{-1} i\Omega\right]$$

$$= \frac{1}{2}\left[\sigma + \Omega\sigma^{-1}\Omega^T\right]$$

Now consider $\alpha = 1/2$ (12)

$$\frac{\rho^{1/2}}{\mathrm{Tr}[\rho^{1/2}]} = \frac{\exp\left[-\frac{1}{2}(\hat{r}-\bar{r})^T H_{(1/2)} (\hat{r}-\bar{r})\right]}{\sqrt{\mathrm{Det}\left(\frac{\sigma_{(1/2)} + i\Omega}{2}\right)}}$$

$$H_{(1/2)} = 1/2\, H$$

for $\alpha = 1/2$

$$\frac{(1+1/x)^\alpha + (1-1/x)^\alpha}{(1+1/x)^\alpha - (1-1/x)^\alpha} = \left(1 + \sqrt{1 - 1/x^2}\right) x$$

$\Rightarrow$

$$\sigma_{(1/2)} = \left(I + \sqrt{I - (\sigma i \Omega)^{-2}}\right)(\sigma i \Omega)\, i\Omega$$

$$= \left(\sqrt{I + (\sigma\Omega)^{-2}} + I\right)\sigma$$

$$\Rightarrow \mathrm{Tr}[\rho^{1/2}] = \frac{1}{\left[\mathrm{Det}\left(\frac{\sigma + i\Omega}{2}\right)\right]^{1/4}} \sqrt{\mathrm{Det}\left(\frac{\left(\sqrt{I + (\sigma\Omega)^{-2}} + I\right)\sigma + i\Omega}{2}\right)}$$

$$\Rightarrow \frac{\rho^2}{\mathrm{Tr}[\rho^2]}$$

$$= \frac{\exp\left[-\frac{1}{2}(\hat{r}-\bar{r})^T 2H (\hat{r}-\bar{r})\right]}{\sqrt{\mathrm{Det}\left(\frac{\frac{1}{2}(\sigma+\Omega\sigma^{-1}\Omega^T)+i\Omega}{2}\right)}}$$

$\Rightarrow$ another expression ~~for~~ for purity is

$$\mathrm{Tr}[\rho^2]$$

$$= \frac{1}{\mathrm{Det}\left(\frac{\sigma+i\Omega}{2}\right)} \sqrt{\mathrm{Det}\left(\frac{\frac{1}{2}(\sigma+\Omega\sigma^{-1}\Omega^T)+i\Omega}{2}\right)}$$

but can show that this collapses to

$$\mathrm{Tr}[\rho^2] = \frac{1}{\sqrt{\mathrm{Det}(\sigma)}}$$ (so much simpler this way)