Lecture 10

Previously, we proved that the uncertainty relation

$$\sigma + i\Omega \geq 0$$

holds for any n-mode state w/ a covariance matrix σ.

Now armed w/ the Williamson decomposition theorem, we can understand this in a different way. Let S be the symplectic matrix diagonalizing σ as

$$S \sigma S^T = D = \bigoplus_{j=1}^{n} d_j \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

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$\sigma + i\Omega \geq 0$

$$\Rightarrow \quad S(\sigma + i\Omega)S^T \geq 0$$

$$\Rightarrow \quad S\sigma S^T + iS\Omega S^T \geq 0$$

$$\Rightarrow \quad D + i\Omega \geq 0$$

$$\Leftrightarrow \quad \bigoplus_{j=1}^{n} d_j \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + i \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \geq 0$$

$$\Leftrightarrow \quad \bigoplus_{j=1}^{n} \begin{bmatrix} d_j & i \\ -i & d_j \end{bmatrix} \geq 0$$

$$\Rightarrow \quad \begin{bmatrix} d_j & i \\ -i & d_j \end{bmatrix} \geq 0 \quad \forall j$$

eigenvalues are

d_j+1, d_j-1 & condition

implies that $d_j \geq 1 \quad \forall j$

If we have a Gaussian state ρ_A characterized by a mean vector $\bar{r}_A$ a covariance matrix σ_A

& another Gaussian state ρ_B w/ mean vector $\bar{r}_B$ & covariance matrix σ_B

then the mean vector of the product state

$\rho_A \otimes \rho_B$ is $\begin{bmatrix} \bar{r}_A \\ \bar{r}_B \end{bmatrix}$

& the covariance matrix is

$$\sigma_A \oplus \sigma_B = \begin{bmatrix} \sigma_A & 0 \\ 0 & \sigma_B \end{bmatrix}$$

One can also run this backwards: if mean vector is $\begin{bmatrix} \bar{r}_A \\ \bar{r}_B \end{bmatrix}$ & cm is $\begin{bmatrix} \sigma_A & 0 \\ 0 & \sigma_B \end{bmatrix}$ then Gaussian state is product.

Purifying Gaussian states to a Gaussian purification

Given a thermal state of a single mode, we can write it as

$$\theta(\bar{n}) = \frac{1}{\bar{n}+1} \sum_{n=0}^{\infty} \left(\frac{\bar{n}}{\bar{n}+1}\right)^n |n\rangle\langle n|$$

where $\bar{n}$ is the mean photon number. Alternatively,

$$\theta(\bar{n}) = \frac{1}{Z(\lambda)} \sum_{n=0}^{\infty} e^{-\lambda(n+\frac{1}{2})} |n\rangle\langle n|$$

for $\lambda > 0$

where $Z(\lambda) = \{e^{\lambda/2} - e^{-\lambda/2}\}^{-1}$

purification of this state is given by two-mode squeezed vacuum state:

$$|\psi(\bar{n})\rangle_{RA} = \frac{1}{\sqrt{\bar{n}+1}} \sum_{n=0}^{\infty} \sqrt{\left(\frac{\bar{n}}{\bar{n}+1}\right)^n} |n\rangle_R |n\rangle_A .$$

Covariance matrix of TMSV

$$\begin{bmatrix} 2\bar{n}+1 & 0 & 2\sqrt{\bar{n}(\bar{n}+1)} & 0 \\ 0 & 2\bar{n}+1 & 0 & -2\sqrt{\bar{n}(\bar{n}+1)} \\ 2\sqrt{\bar{n}(\bar{n}+1)} & 0 & 2\bar{n}+1 & 0 \\ 0 & -2\sqrt{\bar{n}(\bar{n}+1)} & 0 & 2\bar{n}+1 \end{bmatrix} = \begin{bmatrix} (2\bar{n}+1)I & 2\sqrt{\bar{n}(\bar{n}+1)}\sigma_Z \\ 2\sqrt{\bar{n}(\bar{n}+1)}\sigma_Z & (2\bar{n}+1)I \end{bmatrix}$$

By Williamson theorem, any n-mode Gaussian state can be written as

$$\hat{D}_{-\bar{r}}\, \hat{S} \left[\bigotimes_{j=1}^{n} \theta_{A_j}(\bar{n}_j) \right] \hat{S}^{+}\, \hat{D}_{\bar{r}}$$

where $\hat{S}$ is unitary generated by quadratic Hamiltonian.

Then a purification is given by

$$\left[\hat{D}_{-r}\, \hat{S}\right]_{A^n} \bigotimes_{j=1}^{n} |\psi(\bar{n}_j)\rangle_{R_j A_j}$$

for purification

Then mean vector is $\begin{bmatrix} \bar{r} \\ 0 \end{bmatrix}$

+ covariance matrix is

$$\begin{array}{c} A_1 \\ \vdots \\ A_n \\ R_1 \\ \vdots \\ R_n \end{array} \begin{bmatrix} \sigma & S \underbrace{\bigoplus_{j=1}^{n} 2\sqrt{\bar{n}_j(\bar{n}_j+1)}\, \sigma_z} \\ (\bigoplus \cdots)^T S^T & \bigoplus_{j=1}^{n} (2\bar{n}_j+1) I_2 \end{bmatrix}$$

One can arrive at this conclusion from the fact that

$$\sigma = S\left(\bigoplus_{j=1}^{n}(2\bar{n}_j+1)\, I_2\right) S^T$$

covariance matrix for $\bigotimes_{j=1}^{n} |\psi(\bar{n}_j)\rangle_{R_j A_j}$ is

$$\begin{array}{c} A_1 \\ \vdots \\ A_n \\ R_1 \\ \vdots \\ R_n \end{array}\begin{bmatrix} \bigoplus_{j=1}^{n}(2\bar{n}_j+1)\, I_2 & \bigoplus_{j=1}^{n} 2\sqrt{\bar{n}_j(\bar{n}_j+1)}\,\sigma_Z \\ \bigoplus_{j=1}^{n} 2\sqrt{\bar{n}_j(\bar{n}_j+1)}\,\sigma_Z & \bigoplus_{j=1}^{n}(2\bar{n}_j+1)\, I_2 \end{bmatrix}$$

$\downarrow$ symplectic matrix for unitary evolution $\hat{S}_{A^n} \otimes I_{R^n}$ is

$$\begin{bmatrix} S & 0 \\ 0 & I \end{bmatrix}$$

The purity of a state ρ is defined as

$$\mathrm{Tr}[\rho^2]$$

It is less than or equal to one

b/c

$$\rho = \sum_x \lambda_x |\phi_x\rangle\langle\phi_x|$$

$$\Rightarrow \mathrm{Tr}[\rho^2] = \sum_x \lambda_x^2$$

& since $\lambda_x \leq 1 \Rightarrow \lambda_x^2 \leq 1$

& $\sum_x \lambda_x = 1 \Rightarrow \sum_x \lambda_x^2 \leq 1$

If a state is pure, then $\mathrm{Tr}[\rho^2] = 1$

If $\mathrm{Tr}[\rho^2] = 1$ then state is pure.

to see this consider that

$$1 = \mathrm{Tr}[\rho^2] = \sum_x \lambda_x^2$$

$$= \mathrm{Tr}[\rho] = \sum_x \lambda_x \Rightarrow \mathrm{Tr}[\rho]^2 = 1$$

$$= \sum_{x,y} \lambda_x \lambda_y = 1$$

$$\Rightarrow 0 = \mathrm{Tr}[\rho^2] - \mathrm{Tr}[\rho]^2$$

$$= \sum_x f_x^2 - \left[\sum_{x,y} f_x f_y\right]$$

$$= \sum_x f_x^2 - \left[\sum_x f_x^2 + \sum_{x\neq y} f_x f_y\right]$$

$$= \sum_{x\neq y} f_x f_y$$

$\Rightarrow$ it cannot be the case that $f_x f_y > 0$ for any two distinct x & y

$\Rightarrow$ only possibility is that $f_x = 1$ & $f_y = 0 \ \forall y \neq x$.

Goal now is to calculate purity for Gaussian states

Recall decomposition of Gaussian state as

$$\rho = \hat{D}_{-\bar{r}}\, \hat{S} \left(\bigotimes_{j=1}^{n} \theta(\bar{n}_j) \right) \hat{S}^{+} \hat{D}_{\bar{r}}$$

Purity is unitarily invariant

$$\Rightarrow \quad \mathrm{Tr}[\rho^2] = \prod_{j=1}^{n} \mathrm{Tr}[\Theta^2(\bar{n}_j)]$$

Then $\mathrm{Tr}[\Theta^2(\bar{n}_j)]$

$$= \frac{1}{(\bar{n}_j+1)^2} \sum_{n=0}^{\infty} \left(\frac{\bar{n}_j}{\bar{n}_j+1}\right)^{2n}$$

$$= \frac{1}{(\bar{n}_j+1)^2} \frac{1}{1-\left(\frac{\bar{n}_j}{\bar{n}_j+1}\right)^2} = \frac{1}{(\bar{n}_j+1)^2 - \bar{n}_j^2}$$

$$= \frac{1}{2\bar{n}_j+1} = \frac{1}{r_j}$$

$\leftarrow$ symplectic eigenvalue of thermal state

$$\Rightarrow \quad \mathrm{Tr}[\rho^2] = \prod_{j=1}^{n} \frac{1}{r_j}$$

$$= \sqrt{\prod_{j=1}^{n} \frac{1}{r_j^2}}$$

$$= \frac{1}{\sqrt{\prod_{j=1}^{n} r_j^2}} = \frac{1}{\sqrt{\mathrm{Det}(D)}} = \frac{1}{\sqrt{\mathrm{Det}(\sigma)}}$$

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$\Rightarrow$ Purity of Gaussian state is $\mathrm{Tr}[\rho^2] = \frac{1}{\sqrt{\mathrm{Det}(\sigma)}}$

using that $\sigma = \underbrace{S D S^T}_{\text{symp. dec.}}$

& $\mathrm{Det}(S) = 1$.

$\Rightarrow$ Gaussian state is pure iff

$\mathrm{Det}(\sigma) = 1$

Since symp. eig.'s are ≥ 1,

equivalent condition is that

all symplectic eigenvalues

~~can be written as~~ are equal

to one, so that

symplectic decomposition is

given by $\sigma = S S^T$

(i.e., $D = I$).

Computing entropies of a quantum state:

$$S(\rho) = -\mathrm{Tr}[\rho \log \rho]$$

one way to do this is to use unitary invariance of entropy, its additivity, & its value for thermal states.

For a thermal state,

$$\theta(\bar{n}) = \frac{1}{\bar{n}+1} \sum_{n=0}^{\infty} \left(\frac{\bar{n}}{\bar{n}+1}\right)^{n} |n\rangle\langle n|$$

$$= \frac{1}{\bar{n}+1} \left(\frac{\bar{n}}{\bar{n}+1}\right)^{\hat{n}}$$

then $-\mathrm{Tr}[\rho \log \rho]$

$$= -\mathrm{Tr}\left[\cancel{\frac{1}{\bar{n}+1}\left(\frac{\bar{n}}{\bar{n}+1}\right)^{\hat{n}}} \, \theta(\bar{n}) \log \theta(\bar{n}) \right]$$

$$= -\mathrm{Tr}\left[\theta(\bar{n}) \log \frac{1}{\bar{n}+1} \left(\frac{\bar{n}}{\bar{n}+1}\right)^{\hat{n}} \right]$$

$$= -\mathrm{Tr}\left\{\theta(\bar{n}) \log\left(\frac{1}{\bar{n}+1}\right)\right\}$$

$$- \mathrm{Tr}\left\{\theta(\bar{n})\, \hat{n} \log\left(\frac{\bar{n}}{\bar{n}+1}\right)\right\}$$

$$= \log(\bar{n}+1) - \log\left(\frac{\bar{n}}{\bar{n}+1}\right) \mathrm{Tr}\left\{\theta(\bar{n})\hat{n}\right\}$$

$$= \log(\bar{n}+1) - \log\left(\frac{\bar{n}}{\bar{n}+1}\right) \bar{n}$$

$$= (\bar{n}+1)\log(\bar{n}+1) - \bar{n}\log\bar{n}$$

$$\equiv g(\bar{n})$$

then for a general Gaussian state,

~~it follows that~~

$$\rho = \hat{D}_{-\bar{r}}\, \hat{S} \left(\bigotimes_{j=1}^{n} \theta(\bar{n}_j)\right) \hat{S}^{+} \hat{D}_{\bar{r}}$$

it follows that

$$S(\rho) = S\left(\bigotimes_{j=1}^{n} \theta(\bar{n}_j)\right)$$

$$= \sum_{j=1}^{n} S(\theta(\bar{n}_j))$$

in terms of symp. eigenvalues

$$= \sum_{j=1}^{n} g(\bar{n}_j) = \sum_{j=1}^{n} g\left(\frac{r_j - 1}{2}\right)$$

There is a different way to get a formula, which is interesting:
for faithful Gaussian states

let $$\rho = \frac{1}{\sqrt{\mathrm{Det}\left(\frac{\sigma + i\Lambda}{2}\right)}} e^{-\frac{1}{2}(\hat{r}-\bar{r})^T H (\hat{r}-\bar{r})}$$

$$= \hat{D}_{-\bar{r}} \underbrace{\left[\frac{e^{-\frac{1}{2}\hat{r}^T H \hat{r}}}{\sqrt{\mathrm{Det}\left(\frac{\sigma+i\Lambda}{2}\right)}} \right]}_{\rho_0} \hat{D}_{\bar{r}}$$

then $$S(\rho) = S(\rho_0)$$

$$= -\mathrm{Tr}[\rho_0 \log \rho_0]$$

$$= -\mathrm{Tr}\left[\rho_0 \log \frac{e^{-\frac{1}{2}\hat{r}^T H \hat{r}}}{\sqrt{\mathrm{Det}\left(\frac{\sigma+i\Lambda}{2}\right)}}\right]$$

$$= -\mathrm{Tr}\left[\rho_0 \log \frac{1}{\sqrt{\mathrm{Det}\left(\frac{\sigma+i\Lambda}{2}\right)}}\right]$$

$$- \mathrm{Tr}\left[\rho_0 \log e^{-\frac{1}{2}\hat{r}^T H \hat{r}}\right]$$

$$= \frac{1}{2}\log \mathrm{Det}\left(\frac{\sigma+i\Lambda}{2}\right) + \frac{1}{2}\mathrm{Tr}\left[\rho_0 \hat{r}^T H \hat{r}\right]$$

Then $\mathrm{Tr}[\rho_0 \hat{r}^T H \hat{r}]$

$$= \mathrm{Tr}\Big[\rho_0 \sum_{jk} \hat{r}_j H_{jk} \hat{r}_k\Big]$$

$$= \sum_{jk} H_{jk} \, \mathrm{Tr}[\rho_0 \hat{r}_j \hat{r}_k]$$

$$= \frac{1}{2}\sum_{jk} H_{jk} \, \mathrm{Tr}\Big[\rho_0\big(\{\hat{r}_j, \hat{r}_k\} + [\hat{r}_j, \hat{r}_k]\big)\Big]$$

$$= \frac{1}{2}\sum_{jk} H_{jk}\left(\sigma_{jk} + i\Omega_{jk}\right)$$

$$= \frac{1}{2}\sum_{jk} H_{jk}\sigma_{kj} - \frac{i}{2}\sum_{jk} H_{jk}\Omega_{kj}$$

$$= \frac{1}{2}\mathrm{Tr}[H\sigma] - \frac{i}{2}\mathrm{Tr}[H\Omega]$$

$$= \frac{1}{2}\mathrm{Tr}[H\sigma]$$

H sym. & Ω anti. & trace invariant under transpose

$$\Rightarrow \quad S(\rho) = \frac{1}{2}\log \mathrm{Det}\left(\frac{\sigma + i\Omega}{2}\right) + \frac{1}{4}\mathrm{Tr}[H\sigma]$$

can write entirely in terms of σ via $H = 2\,\mathrm{arccoth}(i\Omega\sigma)\, i\Omega$

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So that

$$S(\rho) = \frac{1}{2} \log \mathrm{Det}\left(\frac{\sigma + i\Omega}{2}\right)$$

$$+ \frac{1}{2} \mathrm{Tr}\left[\mathrm{arcoth}(i\Omega\sigma)\, i\Omega\sigma\right]$$