Lecture 9

Last time:

defined a faithful Gaussian state to be a thermal state

$$\frac{e^{-\beta \hat{H}}}{\mathrm{Tr}[e^{-\beta \hat{H}}]} \quad \text{for } \beta > 0$$

of a quadratic Hamiltonian

$$\hat{H} = \frac{1}{2} \hat{r}^T H \hat{r} + \hat{r}^T \bar{r}'$$

where $\bar{r}' \in \mathbb{R}^{2n}$ & H is a $2n \times 2n$ positive definite real matrix.

We can equivalently consider Hamiltonian to have the form

$$\hat{H} = \frac{1}{2} (\hat{r} - \bar{r})^T H (\hat{r} - \bar{r})$$

where $\bar{r} \in \mathbb{R}^{2n}$ & H is $2n \times 2n$ positive definite real matrix.

We showed how to build up a faithful Gaussian state w/ Hamiltonian matrix

$$H' = S^T H S$$

for $H = \bigoplus_{j=1}^{n} \lambda_j \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad \lambda_j > 0 \ \forall j \in \{1, \ldots, n\}$

S symplectic,

& Hamiltonian operator $\hat{H}' = \frac{1}{2}(\hat{r} - \bar{r})^T S^T H S (\hat{r} - \bar{r})$

$$= \frac{1}{2}(\hat{r} - \bar{r})^T H' (\hat{r} - \bar{r})$$

gives density operator

$$\rho_G = \frac{e^{-\frac{1}{2}(\hat{r} - \bar{r})^T H' (\hat{r} - \bar{r})}}{\sqrt{\mathrm{Det}\left(\frac{\sigma' + i\Omega}{2}\right)}} \quad (*)$$

where mean vector of ρ_G is $\bar{r}$

& covariance matrix is

$$\sigma' = S^{-1} \sigma S^{-T} \quad \text{w/} \quad \sigma = \bigoplus_{j=1}^{n} \coth\left(\frac{\lambda_j}{2}\right) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Also, we noted that ρ_G can be written as

$$\rho_G = \frac{\hat{D}_{-\bar{r}}\,\hat{S}\; e^{-\frac{1}{2}\hat{r}^T H \hat{r}}\,\hat{S}^{+}\,\hat{D}_{\bar{r}}}{\sqrt{\mathrm{Det}\left(\frac{\sigma' + i\Omega}{2}\right)}} \quad (\ast\ast)$$

where $\hat{S} = e^{\frac{i}{2}\hat{r}^T \Omega^T \ln S\, \hat{r}}$ (note that for some S we might need two quadratic evolutions)

& $\hat{D}_{\bar{r}} = \exp(i\,\bar{r}^T \Omega\, \hat{r})$

is displacement op.

we also noted that since H is diagonal,

$$\frac{e^{-\frac{1}{2}\hat{r}^T H \hat{r}}}{\sqrt{\mathrm{Det}\left(\frac{\sigma' + i\Omega}{2}\right)}} = \bigotimes_{j=1}^{n} \frac{e^{-\Omega_j(\hat{n} + \frac{1}{2})}}{z(\Omega_j)}$$

$$\text{w/ } z(\Omega_j) = \left[e^{\Omega_j/2} - e^{-\Omega_j/2}\right]^{-1}$$

Note that $\dfrac{e^{-\Omega(\hat{n}+\frac{1}{2})}}{z(\Omega)}$ is typically called bosonic thermal state

& has mean photon number $\langle\hat{n}\rangle = \frac{1}{2}\langle\hat{x}^2 + \hat{p}^2 - 1\rangle$

$= \coth(\Omega/2) - 1/2$

Now we prove, perhaps surprisingly, that the form in (*) is the most general form that a faithful Gaussian state can take.

That is suppose that the faithful Gaussian state is given by

$$\frac{e^{-\frac{1}{2}(\hat{r}-\bar{r})^T H (\hat{r}-\bar{r})}}{\mathrm{Tr}[\,''\,]},$$

where $\bar{r} \in \mathbb{R}^{2n}$ & H is $2n \times 2n$ real positive definite ~~defn~~ matrix.

Then there exists symplectic S such that

$$H = S^T H_{diag} S$$

where $H_{diag} = \bigoplus_{j=1}^{n} \lambda_j \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ for $\lambda_j > 0 \ \forall_{j \in \{1,\dots,n\}}$

then we can write state as in (**).

So it suffices to focus on Hamiltonians of the form

$$\hat{H}' = \frac{1}{2}(\hat{r}-\bar{r})^T H (\hat{r}-\bar{r})$$

when defining faithful Gaussian states.

Consider that

$$\hat{H}' = \frac{1}{2}\sum_{jk}(\hat{r}_j - \bar{r}_j) H_{jk} (\hat{r}_k - \bar{r}_k)$$

$$= \frac{1}{2}\sum_{jk} H_{jk} (\hat{r}_j - \bar{r}_j)(\hat{r}_k - \bar{r}_k)$$

go to 2a

Last time we showed that

$$\hat{D}_{-\bar{r}}\, \hat{r}\, \hat{D}_{\bar{r}} = \hat{r} - \bar{r}$$

where

$$\hat{D}_{\bar{r}} = \exp(i\, \bar{r}^T \Omega \hat{r})$$

So this means that, upon substituting,

$$\hat{H}' = \frac{1}{2}\sum_{jk} H_{jk} (\hat{D}_{-\bar{r}}\, \hat{r}_j\, \hat{D}_{\bar{r}})(\hat{D}_{-\bar{r}}\, \hat{r}_k\, \hat{D}_{\bar{r}})$$

$$= \frac{1}{2}\sum_{jk} H_{jk}\, \hat{D}_{-\bar{r}}\, \hat{r}_j\, \hat{r}_k\, \hat{D}_{\bar{r}}$$

$$= \hat{D}_{\bar{r}}^{\dagger} \left[\frac{1}{2} \, \hat{r}^T H \, \hat{r} \right] \hat{D}_{\bar{r}}$$

Thus we can realize the shift in quadrature operators of Hamiltonian $\hat{H}$ via a unitary displacement operator

It is then natural to consider computing the mean vector, covariance matrix, & normalization of the Gaussian state

$$\frac{e^{-\beta \hat{H}}}{\mathrm{Tr}[e^{-\beta \hat{H}}]} \quad (\text{take } \beta = 1 \text{ WLOG})$$

as a function of the Hamiltonian matrix H & the offset $\bar{r}$.

Before doing so, we establish an important theorem, called the Williamson decomposition theorem.

Williamson

Given a $2n \times 2n$ positive definite real matrix M, $\exists$ a symplectic transformation S (i.e., $S \Omega S^T = \Omega$) such that

$$SMS^T = D \quad w/ \quad D = \bigoplus_{j=1}^{n} d_j \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\& \; d_j > 0 \quad \forall j \in \{1, \dots, n\}$$

the set $\{d_j\}_{j=1}^{n}$ is the set of symplectic eigenvalues of M.

Before proving this, we recall a standard lemma about decomposing real antisymmetric matrices.

<u>Lemma:</u> Let A be a real, full-rank, antisymmetric $2n \times 2n$ matrix.

Then $\exists$ a real orthogonal $2n \times 2n$ matrix O such that

$$O A O^T = \bigoplus_{j=1}^{n} \frac{1}{b_j} \Lambda_1$$

where $\Lambda_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

& b_j [illegible] > 0.

<u>Proof:</u>

Assumptions imply that

A^2 is symmetric & positive definite. Thus $\exists$ an orthogonal transformation O' such that

$$O' A^2 O'^T = B \quad \text{w/} \quad B \text{ diagonal}$$

Let $|\psi\rangle$ be some eigenvector of A^2 w/ eigenvalue $b_1^2 > 0$

then $|\psi'\rangle = \frac{A|\psi\rangle}{b_1}$ is normalized & orthogonal to $|\psi\rangle$

b/c $$\langle\psi|\psi'\rangle = \frac{\langle\psi|A|\psi\rangle}{b_1} = \frac{(\langle\psi|A|\psi\rangle)^T}{b_1}$$

$$= \frac{\langle\psi|A^T|\psi\rangle}{b_1} = -\frac{\langle\psi|A|\psi\rangle}{b_1} = 0.$$

Suppose that $|\varphi\rangle$ is in the subspace orthogonal to span $\{|\psi\rangle, |\psi'\rangle\}$

Then $\langle\varphi|A\psi\rangle = \langle\varphi|\psi'\rangle b_1 = 0$

& $\langle\varphi|A\psi'\rangle = \dfrac{\langle\varphi|A^2\psi\rangle}{b_1} = \dfrac{\langle\varphi|\psi\rangle b_1^2}{b_1}$

$= \langle\varphi|\psi\rangle \cdot b_1$

$= 0$

Furthermore, $\langle\psi|A\psi\rangle = 0 = \langle\psi'|A\psi'\rangle$ b/c A is antisymmetric

Also, $\langle\psi|A\psi'\rangle = \dfrac{\langle\psi|A^2\psi\rangle}{b_1} = \dfrac{b_1^2\langle\psi|\psi\rangle}{b_1}$

$= b_1$

& $\langle\psi'|A\psi\rangle = -b_1$ due to antisymmetry.

this means that

$$O_1^T A \overbrace{\begin{bmatrix} |\psi\rangle & |\psi'\rangle & |v_1\rangle & \cdots & |v_{2n-2}\rangle \end{bmatrix}}^{=O_1}$$

$$= \begin{bmatrix} b_1 \Omega_1 & 0 \\ 0 & A' \end{bmatrix}$$

so this gives 1st step of decomposition. A' is antisymmetric & so repeat exhaustively.

can take $b_1 > 0$ WLOG

b/c $\begin{bmatrix} 0 & -b_1 \\ b_1 & 0 \end{bmatrix}$ related to $\begin{bmatrix} 0 & b_1 \\ -b_1 & 0 \end{bmatrix}$ by orthogonal similarity transform $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Proof of Williamson theorem:

Consider the matrix

$M^{-1/2} \Omega M^{-1/2}$ It is real $2n \times 2n$

Due to symmetry of M & antisymmetry of Ω, it follows that

$M^{-1/2} \Omega M^{-1/2}$ is antisymmetric.

Also full rank b/c M + Ω are.

then $\exists$ real orthogonal transform such that

$$O M^{-1/2} \Omega M^{-1/2} O^T = \bigoplus_{j=1}^{n} d_j^{-1} \Omega_1$$

w/ $d_j > 0$.

Define $\underline{D} = \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_j \end{bmatrix}$ $(n \times n)$

& note that $D = \underline{D} \otimes I_2$

then $\bigoplus_{j=1}^{n} \frac{1}{d_j} \Omega_1 = \underline{D}^{-1} \otimes \Omega_1$

Since

$$D^{1/2} \bigoplus_{j=1}^{n} \frac{1}{d_j} \Lambda_1 D^{1/2}$$

$$= (\underline{D}^{1/2} \otimes I_2)(\underline{D}^{-1} \otimes \Lambda_1)(\underline{D}^{1/2} \otimes I_2)$$

$$= I_n \otimes \Lambda_1 = \Lambda$$

$$\Rightarrow \quad D^{1/2} O M^{-1/2} \Lambda M^{-1/2} O^T D^{1/2} = \Lambda$$

Set $S = D^{1/2} O M^{-1/2}$

$$\Rightarrow \quad S \Lambda S^T = \Lambda$$

& S is symplectic.

Also $S M S^T$

$$= (D^{1/2} O M^{-1/2}) M (D^{1/2} O M^{-1/2})^T$$

$$= D^{1/2} O M^{-1/2} M M^{-1/2} O^T D^{1/2}$$

$$= D^{1/2} O O^T D^{1/2}$$

$$= D^{1/2} D^{1/2} = D \quad .$$

We can now apply the Williamson theorem to the Hamiltonian matrix for a faithful Gaussian state.

Recall that we're considering states of the form

$$\frac{e^{-\hat{H}}}{\mathrm{Tr}[e^{-\hat{H}}]} = \frac{e^{-\frac{1}{2}(\hat{r}-\bar{r})^T H (\hat{r}-\bar{r})}}{\mathrm{Tr}[e^{-\hat{H}}]}$$

$$= \frac{\hat{D}_{-\bar{r}}\, e^{-\frac{1}{2}\hat{r}^T H \hat{r}}\, \hat{D}_{\bar{r}}}{\mathrm{Tr}\left[e^{-\frac{1}{2}\hat{r}^T H \hat{r}}\right]}$$

Using the symplectic diagonalization of H as $H = S_H^T(\Lambda \otimes I_2) S_H$

where $\Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$

& S_H is transpose & inverse of the symplectic transformation that puts H in symplectic normal

Then $\frac{1}{2}\hat{r}^T H \hat{r} = \frac{1}{2}\hat{r}^T S_H^T (\Omega \otimes I_2) S_H \hat{r}$

$$= \frac{1}{2}(S_H \hat{r})^T (\Omega \otimes I_2) S_H \hat{r}$$

we can then think of S_H as a coordinate transformation & can ~~define~~ define new set of quadrature operators as $\hat{r}' = S_H \hat{r}$.
since $[\hat{r}', \hat{r}'^T] = i\Omega$

then the Hamiltonian

$$\frac{1}{2}\hat{r}^T H \hat{r} = \frac{1}{2}\hat{r}'^T (\Omega \otimes I_2)\hat{r}'$$

is diagonal wrt this notation.

Now consider that

$$\frac{1}{2}\hat{r}'^T (\Omega \otimes I_2) \hat{r}' = \frac{1}{2}\sum_{jk} \hat{r}'_j (\Omega \otimes I_2)_{jk} \hat{r}'_k$$

$$= \frac{1}{2}\sum_j \Omega_j \left[\hat{x}'^2_j + \hat{p}'^2_j\right]$$

How to compute the symplectic eigenvalues & symplectic matrix S for a given positive definite M?

Use the usual eigendecomposition of

$$i\Omega M$$

Why does this work?

By the Williamson theorem, it follows that

$$M = S D S^T$$

for S symplectic,

$$D = \bigoplus_{j=1}^{n} d_j \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \underline{D} \otimes I_2$$

where $\underline{D} = \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix}$

Then consider that

$$i\Omega M = i\Omega S D S^T$$

$$= i\Omega S (\underline{D} \otimes I_2) S^T$$

Since $S\Lambda S^T = \Lambda$

$\Rightarrow$ $\cancel{S^T \Lambda S = \Lambda}$ ~~(by transpose~~

$$S\Lambda S^T \Lambda = \Lambda\Lambda = -I$$

$$\Rightarrow S\Lambda S^T \Lambda S = -S$$

$$\Rightarrow S^{-1} S \Lambda S^T \Lambda S = -S^{-1} S = -I$$

$$\Rightarrow \Lambda S^T \Lambda S = -I$$

$$\Rightarrow \Lambda^T \Lambda S^T \Lambda S = -\Lambda^T$$

$$\Rightarrow S^T \Lambda S = \Lambda$$

$$\Rightarrow \Lambda S = S^{-T} \Lambda$$

$$\Rightarrow i\Lambda S (\underline{D} \otimes I_2) S^T$$

$$= i S^{-T} \Lambda (\underline{D} \otimes I_2) S^T$$

$$= i S^{-T} (I_n \otimes \Lambda_1)(\underline{D} \otimes I_2) S^T$$

$$= S^{-T} (\underline{D} \otimes i\Lambda_1) S^T \qquad i\Lambda_1 = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} = -\sigma_Y$$

$$= S^{-T} (\underline{D} \otimes -\sigma_Y) S^T$$

$$-\sigma_y = U(-\sigma_z)U^+$$

w/ $U = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}$

$$\Rightarrow \quad = S^{-T}\left(\underline{D} \otimes U(-\sigma_z)U^+\right) S^T$$

$$= \underbrace{S^{-T}(I_n \otimes U)}_{B}\left(\underline{D} \otimes -\sigma_z\right)\underbrace{(I_n \otimes U^+)S^T}_{B^{-1}}$$

Consider that

$$\underline{D} \otimes -\sigma_z = \begin{bmatrix} -d_1 & & & & & & \\ & d_1 & & & & & \\ & & -d_2 & & & & \\ & & & d_2 & & & \\ & & & & \ddots & & \\ & & & & & -d_n & \\ & & & & & & d_n \end{bmatrix}$$

then the above is a diagonalization

$\Rightarrow$ so ordinary diagonalization of $i\Omega M$ gives eigenvalues $\{-d_1, d_1, -d_2, d_2, \ldots, -d_n, d_n\}$, as pairs containing symplectic eigenvalues of M

& eigenvector matrix is $S^{-T}(I_n \otimes U)$ containing symplectic matrix S.

What is the relationship between the Hamiltonian matrix H & the covariance matrix σ for a general Gaussian state?

It is given by

$$\boxed{\begin{aligned} \sigma &= \coth\left(\frac{i\Omega H}{2}\right) i\Omega \\ &\& \\ H &= 2\,\mathrm{arccoth}(i\Omega\sigma)\, i\Omega \end{aligned}}$$

as a generalization of what we found for diagonal case.

to see this, start w/ positive definite Hamiltonian matrix H

Then symplectic diagonalization is

$$H = S^T \bigoplus_{j=1}^{n} \nu_j \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} S = S^T D S$$

then we argued before that covariance matrix is

$$\sigma = S^{-1} \bigoplus_{j=1}^{n} \coth\left(\frac{\nu_j}{2}\right) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} S^{-T}$$

$$= S^{-1} \coth\left(\frac{D}{2}\right) S^{-T}$$

Then consider that, from previous reasoning,

$$\frac{1}{2} i\Lambda H =$$

$$\frac{1}{2} S^{-1}(I_n \otimes U)(D \otimes -\sigma_z)(I_n \otimes U^+) S$$

$$\Rightarrow \coth\left(\frac{i\Lambda H}{2}\right)$$ (functional calculus)

$$= S^{-1}(I_n \otimes U)\left(\coth\left(\frac{D \otimes -\sigma_z}{2}\right)\right)(I_n \otimes U^+) S$$

$$= S^{-1}(I_n \otimes U)\left[\coth\left(\frac{D}{2}\right) \otimes -\sigma_z\right](I_n \otimes U^+) S$$

(coth is odd function)

$$= S^{-1}\left[\coth\left(\frac{D}{2}\right) \otimes -\sigma_y\right] S$$

$$= S^{-1}\left[\coth\left(\frac{D}{2}\right) \otimes i\Lambda_1\right] S$$

$$\cancel{= S^{-1}\left[I_n \otimes i\Lambda_1\right]\left[\coth\left(\frac{D}{2}\right) \otimes I_2\right] S}$$

$$\cancel{= S^{-1} i\Lambda \coth\left(\frac{D}{2}\right) S}$$

$$\cancel{\neq i\Lambda S^{-1} \coth\left(\frac{D}{2}\right) S}$$

$$= S^{-1}\left[\coth\left(\frac{D}{2}\right)\otimes I_2\right]\left[I_n \otimes i\Omega_1\right] S$$

$$= S^{-1}\coth\left(\frac{D}{2}\right) i\Omega S$$

$$= S^{-1}\coth\left(\frac{D}{2}\right) S^{-T} i\Omega$$

$$= \sigma i \Omega$$

$$\Rightarrow \quad \coth\left(\frac{i\Omega H}{2}\right) = \sigma i \Omega$$

$$\Rightarrow \quad \coth\left(\frac{i\Omega H}{2}\right) i\Omega = \sigma i\Omega (i\Omega)$$

$$= \sigma$$

<u>done</u>

Similar proof gives

$$H = 2\operatorname{arccoth}(i\Omega\sigma)\, i\Omega$$