Lecture 8

Gaussian states:

- Previously, we considered the evolution generated by Hamiltonians that are linear or strictly quadratic in the quadrature operators.
- Now, we will consider the thermal states corresponding to quadratic Hamiltonians.

Consider a Hamiltonian of the form

$$\hat{H} = \frac{1}{2} \hat{r}^T H \hat{r} + \hat{r}^T \bar{r}'$$

where $\bar{r}' \in \mathbb{R}^{2n}$ & H is a positive definite $2n \times 2n$ real matrix

A *faithful* Gaussian, n-mode state is

defined as
$$\frac{e^{-\beta \hat{H}}}{\mathrm{Tr}[e^{-\beta \hat{H}}]} \quad \text{for } \beta > 0$$

faithful means that the state is positive definite (has full support)

Consider that

$$\hat{H}' = \frac{1}{2}(\hat{r} - \bar{r})^T H (\hat{r} - \bar{r})$$
$$= \frac{1}{2}\left(\hat{r}^T H \hat{r} - 2\bar{r}^T H \hat{r} + \|\bar{r}\|_2^2\right)$$
$$= \frac{1}{2}\hat{r}^T H \hat{r} - \hat{r}^T H \bar{r} + \frac{1}{2}\|\bar{r}\|_2^2$$

Then ~~substituting~~ setting $\bar{r} = -H^{-1} r'$, we recover the original form of the Hamiltonian stated.

Also, when exponentiating $\hat{H}'$ as $e^{-\beta \hat{H}'}$, the constant term $\frac{1}{2}\|\bar{r}\|_2^2$ is eliminated after normalization

2a

It is natural to consider computing the mean vector, covariance matrix, & normalization for a faithful Gaussian state as a function of $\bar{r}$ & H.

Let us start simple. Suppose a single-mode state w/ $H = \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ for $\lambda > 0$ & $\bar{r} = 0$.

Then the state is given by

$$\frac{e^{-\frac{1}{2}\hat{r}^T H \hat{r}}}{\mathrm{Tr}[e^{-1/2\,\hat{r}^T H \hat{r}}]}$$

Consider that

$$\frac{1}{2}\hat{r}^T H \hat{r} = \frac{1}{2}[\hat{x}\ \hat{p}]\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\begin{bmatrix} \hat{x} \\ \hat{p} \end{bmatrix}$$

$$= \frac{\lambda}{2}(\hat{x}^2 + \hat{p}^2)$$

Now use that $\hat{n} = \hat{a}^\dagger \hat{a} = \left(\frac{\hat{x} - i\hat{p}}{\sqrt{2}}\right)\left(\frac{\hat{x} + i\hat{p}}{\sqrt{2}}\right)$

2b

$$= \frac{1}{2}\left[\hat{x}^2 + \hat{p}^2 + i[\hat{x},\hat{p}]\right]$$

$$= \frac{1}{2}\left[\hat{x}^2 + \hat{p}^2 - 1\right] \Rightarrow \frac{\hat{x}^2+\hat{p}^2}{2} = \hat{n} + \frac{1}{2}$$

then $\frac{1}{2}\hat{r}^T H \hat{r} = \lambda\left(\hat{n}+\frac{1}{2}\right)$

Now use that $\hat{n} = \sum_{n=0}^{\infty} n|n\rangle\langle n|$

$\Rightarrow \lambda(\hat{n}+\frac{1}{2}) = \sum_{n=0}^{\infty} \lambda(n+\frac{1}{2})|n\rangle\langle n|$

to write $e^{-1/2\,\hat{r}^T H \hat{r}}$ as

$$\Rightarrow e^{-\frac{1}{2}\hat{r}^T H \hat{r}} = \sum_{n=0}^{\infty} e^{-\lambda(n+\frac{1}{2})}|n\rangle\langle n|$$

$$= e^{-\lambda/2}\sum_{n=0}^{\infty} e^{-\lambda n}|n\rangle\langle n|$$

$$\Rightarrow \mathrm{Tr}\left[e^{-\frac{1}{2}\hat{r}^T H \hat{r}}\right] = e^{-\lambda/2}\sum_{n=0}^{\infty} e^{-\lambda n}$$

$$= e^{-\lambda/2}\frac{1}{1-e^{-\lambda}}$$

$$= \frac{1}{e^{\lambda/2}-e^{-\lambda/2}}$$

$$\equiv z(\lambda)$$

2c

What about the mean vector?

Consider that $\langle n | \hat{x} | n \rangle$

$$= \langle n | \frac{\hat{a} + \hat{a}^{\dagger}}{\sqrt{2}} | n \rangle$$

$$= \frac{1}{\sqrt{2}} \left[\langle n | \hat{a} | n \rangle + \langle n | \hat{a}^{\dagger} | n \rangle \right]$$

$$= \frac{1}{\sqrt{2}} \left[\langle n | \sqrt{n} | n-1 \rangle + \langle n | \sqrt{n+1} | n+1 \rangle \right]$$

$$= 0$$

Similarly, $\langle n | \hat{p} | n \rangle = 0$

so any Fock-diagonal state has mean vector equal to zero & thus

$$\mathrm{Tr}\left[e^{-\frac{1}{2}\hat{r}^T H \hat{r}} \hat{x} \right] = 0 = \mathrm{Tr}\left[e^{-\frac{1}{2}\hat{r}^T H \hat{r}} \hat{p} \right]$$

What about the co-variance matrix?

i.e. $\mathrm{Tr}\left[\{\hat{r}_i, \hat{r}_j\} e^{-\frac{1}{2}\hat{r}^T H \hat{r}} \right]$

can show that $\langle n | \hat{x}\hat{p} + \hat{p}\hat{x} | n \rangle = 0$

& $\langle n | 2\hat{x}^2 | n \rangle = \langle n | 2\hat{p}^2 | n \rangle = 2n + 1$

2d

so then

$$\mathrm{Tr}\left[\frac{e^{-\frac{1}{2}\hat{r}^T H \hat{r}}}{Z(\beta)}\{\hat{x},\hat{p}\}\right] = 0$$

$$+\ 2\,\mathrm{Tr}\left[\hat{x}^2\,\frac{e^{-\frac{1}{2}\hat{r}^T H \hat{r}}}{Z(\beta)}\right]$$

$$= \frac{2}{Z(\beta)}\mathrm{Tr}\left[\hat{x}^2 \sum_{n=0}^{\infty} e^{-\beta(n+\frac{1}{2})}\,|n\rangle\langle n|\right]$$

$$= \frac{2}{Z(\beta)} \sum_{n=0}^{\infty} e^{-\beta(n+\frac{1}{2})}\,\langle n|\hat{x}^2|n\rangle$$

$$= \frac{2}{Z(\beta)} \sum_{n=0}^{\infty} e^{-\beta(n+\frac{1}{2})}\,(n+\tfrac{1}{2})$$

$$= 1 + 2\frac{e^{-\beta/2}}{Z(\beta)} \sum_{n=0}^{\infty} e^{-\beta n}\, n$$

$$= 1 + 2\frac{e^{-\beta/2}}{Z(\beta)}\left[\frac{-d}{d\beta}\left(\sum_{n=0}^{\infty} e^{-\beta n}\right)\right]$$

$$= 1 + 2\frac{e^{-\beta/2}}{Z(\beta)}\left[\frac{-d}{d\beta}\left(\frac{1}{1-e^{-\beta}}\right)\right]$$

$$= \coth\left(\frac{\beta}{2}\right) \equiv \nu(\beta) > 1 \quad \text{for } \beta > 0$$

where $\coth(x) = \dfrac{e^{x}+e^{-x}}{e^{x}-e^{-x}}$

Similarly, $2 \operatorname{Tr}\left[\hat{p}^2 \frac{e^{-\frac{1}{2}\hat{r}^T H \hat{r}}}{Z(\lambda)} \right] = \coth\left(\frac{\lambda}{2}\right)$ 2e

So a single-mode Hamiltonian

$$H = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \quad \text{for } \lambda > 0$$

has mean vector equal to zero,

covariance matrix given by $\begin{bmatrix} v(\lambda) & 0 \\ 0 & v(\lambda) \end{bmatrix}$

where $v(\lambda) = \coth\left(\frac{\lambda}{2}\right) > 1$

+ normalization

$$\operatorname{Tr}\left[e^{-\frac{1}{2}\hat{r}^T H \hat{r}} \right] = Z(\lambda) = \frac{1}{e^{\lambda/2} - e^{-\lambda/2}}$$

If we had instead started by specifying the covariance matrix as $\sigma = \begin{bmatrix} v & 0 \\ 0 & v \end{bmatrix}$

for v satisfying $\begin{bmatrix} v & 0 \\ 0 & v \end{bmatrix} + i \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \geq 0$

$v > 0$ $\iff$ $v + 1 \geq 0,\ v - 1 \geq 0 \Rightarrow v \geq 1$

then the Hamiltonian matrix elements are given by the inverse operation

$$\lambda(v) = 2 \operatorname{arcoth}(v) = 2\operatorname{arcoth}(v) \qquad |x| > 1$$

where $\operatorname{arcoth}(x) = \frac{1}{2} \ln\left(\frac{x+1}{x-1}\right)$

normalization function can be written as (2f)

$$\therefore \hat{z}(\lambda(v)) = \frac{1}{2}\sqrt{v^2-1}$$

$$= \frac{1}{2}\sqrt{\mathrm{Det}\left(\begin{bmatrix} v & i \\ -i & v \end{bmatrix}\right)}$$

$$= \frac{1}{2}\sqrt{\mathrm{Det}(\sigma + i\Omega)}$$

$$= \sqrt{\mathrm{Det}\left(\frac{\sigma+i\Omega}{2}\right)}$$

So this is how we can express the normalization $\mathrm{Tr}\left[e^{-\frac{1}{2}\hat{r}^T H \hat{r}}\right]$ in terms of the covariance matrix σ of the state

$$\frac{e^{-\frac{1}{2}\hat{r}^T H \hat{r}}}{\mathrm{Tr}[\;''\;]}$$

Now consider an n-mode state w/ a diagonal Hamiltonian matrix

$$H = \begin{bmatrix} \lambda_1 & & & & & & \\ & \lambda_1 & & & & 0 & \\ & & \lambda_2 & & & & \\ & & & \lambda_2 & & & \\ & & & & \ddots & & \\ & 0 & & & & \lambda_n & \\ & & & & & & \lambda_n \end{bmatrix} = \bigoplus_{j=1}^{n} \lambda_j \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

w/ $\lambda_j > 0 \;\forall j$

Then the Hamiltonian operator is

$$\frac{1}{2}\hat{r}^T H \hat{r} = \frac{1}{2}\sum_{j=1}^{n} \lambda_j\left(\hat{x}_j^2 + \hat{p}_j^2\right)$$

& so $$e^{-\frac{1}{2}\hat{r}^T H \hat{r}} = e^{-\frac{1}{2}\sum_{j=1}^{n} \lambda_j\left(\hat{x}_j^2 + \hat{p}_j^2\right)}$$

$$= \bigotimes_{j=1}^{n} e^{-\frac{\lambda_j}{2}\left(\hat{x}_j^2 + \hat{p}_j^2\right)}$$

So arguments from last ~~[illegible]~~ calculation apply:

mean vector is equal to zero

Normalization is given by

$$\mathrm{Tr}\left[\bigotimes_{j=1}^{n} e^{-\frac{\lambda_j}{2}(\hat{x}_j^2 + \hat{p}_j^2)}\right]$$

$$= \prod_{j=1}^{n} \mathrm{Tr}\left[e^{-\frac{\lambda_j}{2}(\hat{x}_j^2 + \hat{p}_j^2)}\right]$$

$$= \prod_{j=1}^{n} z(\lambda_j)$$

covariance matrix is diagonal matrix w/

$$\sigma = \bigoplus_{j=1}^{n} \nu(\lambda_j)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{where } \nu(\lambda_j) = \coth\left(\frac{\lambda_j}{2}\right) > 1$$

2h

If covariance matrix elements are given as $\bigoplus_{j=1}^{n} v_j \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \sigma$ w/ $v_j > 1$

$\Rightarrow$ Hamiltonian $\bigoplus_{j=1}^{n} f(v_j) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ for $f(v) = 2\operatorname{arccoth}(v) > 0$

then normalization is

$$\prod_{j=1}^{n} z(f(v_j)) = \prod_{j=1}^{n} \frac{1}{2}\sqrt{v_j^2 - 1}$$

$$= \prod_{j=1}^{n} \sqrt{\mathrm{Det}\left(\frac{\sigma_j + i\Omega_1}{2}\right)}$$

$$= \sqrt{\prod_{j=1}^{n} \mathrm{Det}\left(\frac{\sigma_j + i\Omega_1}{2}\right)}$$

$$= \sqrt{\mathrm{Det}\left(\frac{\sigma + i\Omega}{2}\right)}$$

because

$$\sigma + i\Omega = \bigoplus_{j=1}^{n} \sigma_j + i\Omega_1$$

where $\sigma_j = v_j \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

+ we used the fact that

$$\mathrm{Det}(A \oplus B) = \mathrm{Det}(A) \cdot \mathrm{Det}(B)$$

So for multimode states
w/ Hamiltonian matrix $H = \bigoplus_{j=1}^{n} \lambda_j \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

the state $\dfrac{e^{-\frac{1}{2}\hat{r}^T H \hat{r}}}{\mathrm{Tr}[\,\cdot\cdot\,]}$ has mean vector
equal to zero,
covariance matrix

$$\sigma = \bigoplus_{j=1}^{n} \nu(\lambda_j) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

w/ $\nu(\lambda_j) = \coth\left(\frac{\lambda_j}{2}\right)$

+ normalization
$\prod_{j=1}^{n} Z(\lambda_j)$, which
is equal to

$$\sqrt{\mathrm{Det}\left(\frac{\sigma + i\Omega}{2}\right)}$$

Suppose now that we take such a
diagonal Hamiltonian H +
act on it by congruence w/
a symplectic matrix S, to produce a new
Ham. matrix

$$\Rightarrow \quad H' = S^T H S$$

By earlier results, where $S = e^{\Omega A}$
for symmetric real A

2j

As shown previously

$$S\hat{r} = e^{\Omega A}\hat{r} = e^{\frac{i}{2}\hat{r}^T A \hat{r}}\ \hat{r}\ e^{-\frac{i}{2}\hat{r}^T A \hat{r}}$$

for real, symmetric A

$$\Rightarrow\ S^{-1}\hat{r} = e^{-\Omega A}\hat{r} = e^{\frac{i}{2}\hat{r}^T(-A)\hat{r}}\ \hat{r}\ e^{-\frac{i}{2}\hat{r}^T(-A)\hat{r}}$$

$$= e^{-\frac{i}{2}\hat{r}^T A \hat{r}}\ \hat{r}\ e^{\frac{i}{2}\hat{r}^T A \hat{r}}$$

$$\Rightarrow\ \frac{1}{2}\hat{r}^T S^T H S \hat{r} = \frac{1}{2}(S\hat{r})^T H S\hat{r}$$

$$= \frac{1}{2}\left(e^{\frac{i}{2}\hat{r}^T A \hat{r}}\hat{r}e^{-\frac{i}{2}\hat{r}^T A \hat{r}}\right)^T H \left(e^{\frac{i}{2}\hat{r}^T A \hat{r}}\hat{r}e^{-\frac{i}{2}\hat{r}^T A \hat{r}}\right)$$

$$= e^{\frac{i}{2}\hat{r}^T A \hat{r}}\ \frac{1}{2}\hat{r}^T H \hat{r}\ e^{-\frac{i}{2}\hat{r}^T A \hat{r}}$$

$$\Rightarrow\ e^{-\frac{1}{2}\hat{r}^T S^T H S \hat{r}}$$

$$= e^{\frac{i}{2}\hat{r}^T A \hat{r}}\ e^{-\frac{1}{2}\hat{r}^T H \hat{r}}\ e^{-\frac{i}{2}\hat{r}^T A \hat{r}}$$

$\Rightarrow$ mean vector of $\dfrac{e^{-\frac{1}{2}\hat{r}^T S^T H S \hat{r}}}{\mathrm{Tr}[\,''\,]}$

$$= \mathrm{Tr}\left[\hat{r}\ e^{\frac{i}{2}\hat{r}^T A \hat{r}}\ \frac{e^{-\frac{1}{2}\hat{r}^T H \hat{r}}}{\mathrm{Tr}[\,''\,]}\ e^{-\frac{i}{2}\hat{r}^T A \hat{r}}\right]$$

2k

$$= \mathrm{Tr}\left[e^{-\frac{i}{2}\hat{r}^T A \hat{r}}\, \hat{r}\, e^{\frac{i}{2}\hat{r}^T A \hat{r}}\, \frac{e^{-\frac{1}{2}\hat{r}^T H \hat{r}}}{\mathrm{Tr}[\,''\,]} \right]$$

$$= \mathrm{Tr}\left[S^{-1}\hat{r}\, \frac{e^{-\frac{1}{2}\hat{r}^T H \hat{r}}}{\mathrm{Tr}[\,''\,]} \right]$$

$$= S^{-1}\, \mathrm{Tr}\left[\hat{r}\, \frac{e^{-\frac{1}{2}\hat{r}^T H \hat{r}}}{\mathrm{Tr}[\,''\,]} \right] = S^{-1} 0 = 0$$

What is covariance matrix?

given by (since zero mean)

$$\mathrm{Tr}\left[\{\hat{r}, \hat{r}^T\}\, e^{\frac{i}{2}\hat{r}^T A \hat{r}}\, \frac{e^{-\frac{1}{2}\hat{r}^T H \hat{r}}}{\mathrm{Tr}[\,''\,]}\, e^{-\frac{i}{2}\hat{r}^T A \hat{r}} \right]$$

$$= \mathrm{Tr}\left[e^{-\frac{i}{2}\hat{r}^T A \hat{r}}\, \{\hat{r}, \hat{r}^T\}\, e^{\frac{i}{2}\hat{r}^T A \hat{r}}\, \frac{e^{-\frac{1}{2}\hat{r}^T H \hat{r}}}{\mathrm{Tr}[\,''\,]} \right]$$

$$= \mathrm{Tr}\left[\{S^{-1}\hat{r}, (S^{-1}\hat{r})^T\}\, \frac{e^{-\frac{1}{2}\hat{r}^T H \hat{r}}}{\mathrm{Tr}[\,''\,]} \right]$$

$$= S^{-1}\, \mathrm{Tr}\left[\{\hat{r}, \hat{r}^T\}\, \frac{e^{-\frac{1}{2}\hat{r}^T H \hat{r}}}{\mathrm{Tr}[\,''\,]} \right] S^{-T}$$

2l

$$= S^{-1} \sigma S^{-T} = \sigma'$$

$$\Rightarrow \sigma' = S^{-1}\left(\bigoplus_{j=1}^{n} \coth\left(\frac{\delta_j}{2}\right) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) S^{-T}$$

What about the normalization factor?

$$\mathrm{Tr}\left[e^{-\frac{1}{2}\hat{r}^T H' \hat{r}} \right]$$

$$= \mathrm{Tr}\left[e^{\frac{i}{2}\hat{r}^T A \hat{r}} \, e^{-\frac{1}{2}\hat{r}^T H \hat{r}} \, e^{-\frac{i}{2}\hat{r}^T A \hat{r}} \right]$$

$$= \mathrm{Tr}\left[e^{-\frac{1}{2}\hat{r}^T H \hat{r}} \right] = \sqrt{\mathrm{Det}\left(\frac{\sigma + i\Omega}{2} \right)}$$

Using the fact that

$$\mathrm{Det}(S) = 1 = \mathrm{Det}(S^{-1})$$

for $\qquad = \mathrm{Det}(S^{-T})$

symplectic S, we have that

$$\sqrt{\mathrm{Det}\left(\frac{\sigma + i\Omega}{2}\right)} = \sqrt{\mathrm{Det}(S^{-1})\, \mathrm{Det}\left(\frac{\sigma + i\Omega}{2}\right) \mathrm{Det}(S^{-T})}$$

$$= \sqrt{\mathrm{Det}\left(S^{-1}\left(\frac{\sigma + i\Omega}{2}\right) S^{-T}\right)} = \sqrt{\mathrm{Det}\left(\frac{\sigma' + i\Omega}{2}\right)}$$

$$= \sqrt{\mathrm{Det}\left(\frac{\sigma' + i\Lambda}{2}\right)}$$

where we used that $S_A \Lambda S_A^T = \Lambda$

Now suppose that we act on new state w/ displacement operator

$$\hat{D}_{-\bar{r}} \, e^{-\frac{1}{2}\hat{r}^T H' \hat{r}} \, \hat{D}_{\bar{r}}$$

where $\hat{D}_{\bar{r}} = \exp(i\,\bar{r}^T \Lambda \hat{r})$

This is equal to

$$e^{-\frac{1}{2}\left[\hat{D}_{-\bar{r}}\, \hat{r}^T H' \hat{r}\, \hat{D}_{\bar{r}}\right]}$$

& now write $\hat{r}^T H' \hat{r} = \sum_{jk}' \hat{r}_j H'_{jk} \hat{r}_k$

to see that $\hat{D}_{-\bar{r}}\, \hat{r}^T H' \hat{r}\, \hat{D}_{\bar{r}} =$

$$\sum_{jk}' \hat{D}_{-\bar{r}}\, \hat{r}_j\, \hat{D}_{\bar{r}}\, H'_{jk}\, \hat{D}_{-\bar{r}}\, \hat{r}_k \hat{D}_{\bar{r}}$$

$$= \sum_{jk}' (\hat{r}_j - \bar{r}_j) H'_{jk} (\hat{r}_j - \bar{r}_j)$$

(2n)

$$\Rightarrow e^{-\frac{1}{2}(\hat{r}-\bar{r})^T H' (\hat{r}-\bar{r})}$$

under this change,

mean vector translates from zero to $\bar{r}$.

covariance matrix σ' unchanged, because it is invariant under changes of the mean vector.

Normalization unchanged.

So we can write this faithful Gaussian state as

$$\frac{e^{-\frac{1}{2}(\hat{r}-\bar{r})^T H' (\hat{r}-\bar{r})}}{\sqrt{\operatorname{Det}\left(\frac{\sigma' + i\Omega}{2}\right)}}$$

notice similarity to classical multivariate Gaussian density.

where $H' = S^T \bigoplus_{j=1}^{n} \lambda_j \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} S$

$$\sigma' = S^{-1} \bigoplus_{j=1}^{n} \coth\left(\frac{\lambda_j}{2}\right) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} S^{-T}$$

It turns out that this is the most general form!!

2na

By everything that we've done,

it follows that

$$\frac{e^{-\frac{1}{2}(\hat{r}-\bar{r})^T H'(\hat{r}-\bar{r})}}{\sqrt{\mathrm{Det}\left(\frac{\sigma'+i\Omega}{2}\right)}} = \frac{\hat{D}_{-\bar{r}}\,\hat{S}_A\, e^{-\frac{1}{2}\hat{r}^T H \hat{r}}\,\hat{S}_A^{\dagger}\,\hat{D}_{\bar{r}}}{\mathrm{Tr}\left[e^{-\frac{1}{2}\hat{r}^T H \hat{r}}\right]}$$

where $H = \bigoplus_{j=1}^{n} \lambda_j \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\hat{S}_A = e^{\frac{i}{2}\hat{r}^T A \hat{r}}$

for $\lambda_j > 0$

$$\& \ \mathrm{Tr}\left[e^{-\frac{1}{2}\hat{r}^T H \hat{r}}\right] = \sqrt{\mathrm{Det}\left(\frac{\sigma+i\Omega}{2}\right)}$$

$$= \sqrt{\mathrm{Det}\left(\frac{\sigma'+i\Omega}{2}\right)}$$

w/ $\sigma = \bigoplus_{j=1}^{n} \nu(\lambda_j) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

where $\nu(\lambda) = \coth\left(\frac{\lambda}{2}\right)$

20

Why is the determinant of a symplectic matrix S equal to 1? $(\det(S)=1)$

Consider that $S \Lambda S^T = \Lambda$

$$\Rightarrow \det(S \Lambda S^T) = \det(\Lambda)$$

Since $\det(\Lambda) = 1$

$$\Rightarrow \det(S)\det(\Lambda)\det(S^T) = \det(\Lambda) = 1$$

$$\Rightarrow \det(S)\det(S^T) = 1$$

$$\Rightarrow [\det(S)]^2 = 1$$

So $\det(S) = +1$ or -1

Now eliminate possibility that $\det(S) = -1$

Since any symplectic S is invertible (thus full rank)

it follows that

$S^T S$ is symmetric positive definite

$\Rightarrow$ eigenvalues of $S^T S + I$ are > 1.

Then $S^T S + I = S^T(S + S^{-T})$

$$= S^T(S + \Lambda S \Lambda^T)$$

2p

b/c $S \Lambda S^T = \Lambda \Rightarrow$

$$S \Lambda S^T \Lambda^T = \Lambda \Lambda^T = I$$

$$\Rightarrow S^{-1} = \Lambda S^T \Lambda^T$$

$$\Rightarrow S^{-T} = \Lambda S \Lambda^T$$

Consider that $\Lambda = \bigoplus_{j=1}^{n} \Lambda_1 = I_n \otimes \Lambda_1$

$$\Lambda_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Now write $S = \sum_{j,k \in \{0,1\}} S_{jk} \otimes |j\rangle\langle k|$

$$\Rightarrow S + \Lambda S \Lambda^T$$

$$= \sum_{j,k} S_{jk} \otimes |j\rangle\langle k| + (I_n \otimes \Lambda_1)(S_{jk} \otimes |j\rangle\langle k|)(I_n \otimes \Lambda_1^T)$$

$$= \sum_{j,k} S_{j,k} \otimes \left[|j\rangle\langle k| + \Lambda_1 |j\rangle\langle k| \Lambda_1^T \right]$$

$$\Lambda_1 |0\rangle = -|1\rangle$$

$$\Lambda_1 |1\rangle = |0\rangle$$

$$\Rightarrow |j\rangle\langle k| + \Lambda_1 |j\rangle\langle k| \Lambda_1^T = |j\rangle\langle k| + (-1)^{j \oplus 1} |j \oplus 1\rangle\langle k \oplus 1| \, (-1)^{k \oplus 1}$$

$$= |j\rangle\langle k| + (-1)^{j \oplus k} |j \oplus 1\rangle\langle k \oplus 1|$$

2a

$$\Rightarrow \quad = (S_{00} + S_{11}) \otimes |0\rangle\langle 0| + (S_{01} - S_{10}) \otimes |0\rangle\langle 1|$$

$$+ (-S_{01} + S_{10}) \otimes |1\rangle\langle 0| + (S_{00} + S_{11}) \otimes |1\rangle\langle 1|$$

Define $C = S_{00} + S_{11}$, $D = S_{01} - S_{10}$ } these are real matrices

$$\Rightarrow \quad S + \Lambda S \Lambda^T = C \otimes |0\rangle\langle 0| + D \otimes |0\rangle\langle 1|$$

$$- D \otimes |1\rangle\langle 0| + C \otimes |1\rangle\langle 1|$$

$$= (I \otimes \underline{u}) \left([C + iD] \otimes |0\rangle\langle 0| + [C - iD] \otimes |1\rangle\langle 1| \right)$$

$$(I \otimes \underline{u}^\dagger)$$

where $\underline{u} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}$

then

$$0 < 1 < \det(S^T S + I)$$

$$= \det(S^T (S + \Lambda S \Lambda^T))$$

$$= \det(S^T) \det(S + \Lambda S \Lambda^T)$$

$$= \det(S) \det(I \otimes \underline{u}) \det(C + iD) \det(C - iD) \det(I \otimes \underline{u}^\dagger)$$

$$= \det(S) \det(C + iD) \det(\overline{C + iD})$$

$$= \det(S) \det(C + iD) \overline{\det(C + iD)}$$

2r

$$= \det(S)\ |\det(C+iD)|^2$$

Since $\det(S)\ |\det(C+iD)|^2 > 0$

must be the case that

$|\det(C+iD)|^2 > 0$

can divide through &

conclude that $\det(S) > 0$

$\Rightarrow \det(S) = 1$

by elimination.